The Kaczmarz algorithm for MCAO

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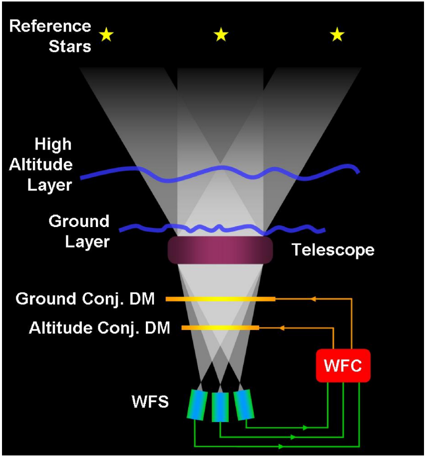
Multi conjugate adaptive optics (MCAO)



(Source: ESO)

The MCAO system utilizes **several guide stars** each assigned with a WFS and **several deformable mirrors**.

Solution consists of three steps: reconstruct incoming wavefront (**SH-WFS**), reconstruct the turbulence profile (**tomography**) and choose best correction (**fitting step**).

Standard approach

- discretize atmosphere into a finite number of layers
- set up system matrix $\mathbf{A}$ that maps sensor measurements to mirror commands

$$\begin{aligned} \mathbf{A} &: \quad (dim\ WFS)^2 \cdot (\#WFS) \times (dim\ DM)^2 \times (\#DMs) \\ TMT &\sim \quad 34.752 \times 7577 \\ E-ELT &\sim \quad 60.480 \times 9.296 \\ Rec.\ time &\sim \quad 2ms, \\ & \qquad ill\ cond.\ system \end{aligned}$$

- Drawbacks:
  1. high computational cost
  2. new system matrix needed for each guide star configuration / fitting
  3. speedup needs transformation to different bases
  4. approach does not use specific properties of the subproblems

3 - Step - Solver

- Alternative: solve the problems sequently!

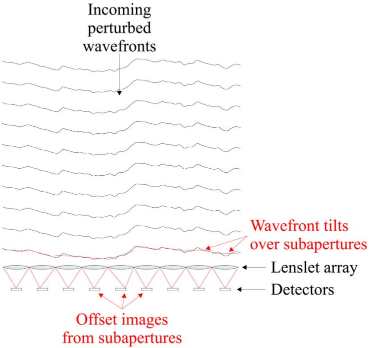
1. Reconstruct incoming wavefronts from SH wavefront measurements
2. use reconstructed wavefronts to determine turbulence layers
3. compute optimal mirror shapes from the reconstructed atmosphere

Can only be successful if all subproblems can be solved extremely fast!

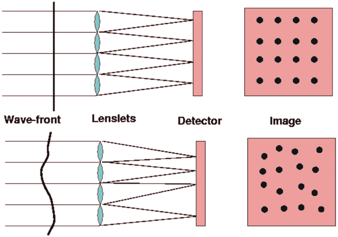
- Advantages

  - higher flexibility (GS configuration, optimization directions)
  - employ specific properties of the subproblem
  - can result in a matrix free approach - no precomputations needed!
  - approach can be used for MOAO (different fitting step)

Measurements from a Shack - Hartmann Sensor



(Source: Wikipedia)



(Source: Tokovinin)

$$(K\Phi)(i,j) := \left(\int_{\Omega_{ij}} \frac{\partial}{\partial x}\Phi(x,y)\,dx\,dy \;, \int_{\Omega_{ij}} \frac{\partial}{\partial y}\Phi(x,y)\,dx\,dy \right)$$

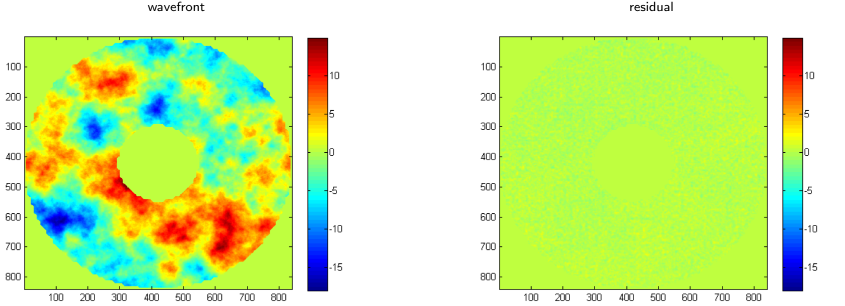
$i, j = 1, \ldots N.$

The Cumulative Reconstructor (CuReD)

- Task: Inversion data from the Shack Hartmann sensor

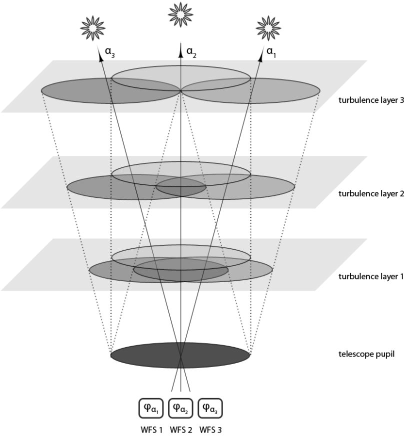
CuRe $\sim$ fast direct reconstructor based on integration

$\longrightarrow$ see talk on CuReD



Reconstruction for a 42m telescope, sensor size 84x84

The tomography problem



- Input: reconstructed wavefronts
- Model assuming geometric propagation:

$$\sum_{l=1}^{L} \Phi^{(l)}(\mathbf{r} + h_l \boldsymbol{\alpha}_g) = \varphi_{\alpha_g}(\mathbf{r})$$

- Task: reconstruct $\Phi^{(l)}$ from wavefronts $\varphi_{\alpha_g}(\mathbf{r})$

$\Longrightarrow$ **ill-posed inverse problem**, requires **regularization**.

MCAO: Notations

- Model for the data:

$$\mathbf{A}_{\boldsymbol{\alpha}_g}\boldsymbol{\Phi} := \sum_{l=1}^{L} \Phi^{(l)}(c_l\mathbf{r} + h_l\boldsymbol{\alpha}_g) = \varphi_{\alpha_g}(\mathbf{r}) \ , \mathbf{r} \in \boldsymbol{\Omega}_D \ .$$

$$c_l := 1 - \frac{h_l}{h_{LGS}}, \qquad h_{LGS}\text{- LGS height}$$

$$\boldsymbol{\Phi} = (\Phi^{(1)}, \cdots, \Phi^{(L)})^T, \qquad \langle \boldsymbol{\Phi}, \boldsymbol{\Psi} \rangle = \sum_{l=1}^{L} \frac{1}{\gamma_l} \langle \Phi^{(l)}, \Psi^{(l)} \rangle_X$$

$X \in \{L_2(\boldsymbol{\Omega}_l), H^1(\boldsymbol{\Omega}_l)\}$

- $\gamma_l \sim$ relative strength of layer l,

$$\sum_{l=1}^{L} \gamma_l = 1$$

incorporates a priori information into the reconstruction

Kaczmarz iteration

- acts on the smaller subsystems $\mathbf{A}_{\boldsymbol{\alpha}_g}\boldsymbol{\Phi} = \varphi_{\alpha_g}$:

Algorithm:

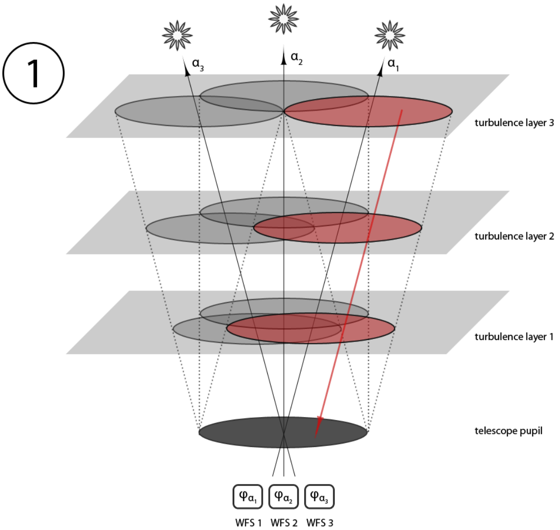
- Choose $\boldsymbol{\Phi}_0$
- For $i = 1, \cdots$
  - $\boldsymbol{\Phi}_{i,0} = \boldsymbol{\Phi}_{i-1}$
  - For $g = 1, \cdots, G$ do

$$\boldsymbol{\Phi}_{i,l} = \boldsymbol{\Phi}_{i,l-1} + \beta_l \mathbf{A}^*_{\boldsymbol{\alpha}_g}\left(\varphi_{\alpha_g} - \mathbf{A}_{\boldsymbol{\alpha}_g}\boldsymbol{\Phi}_{i,l-1}\right)$$

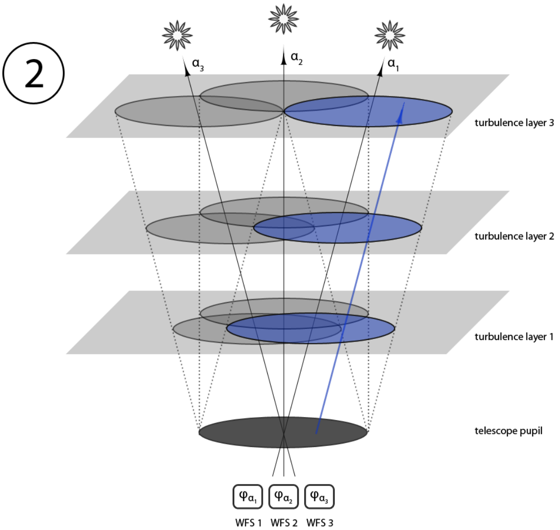
    end
  - $\boldsymbol{\Phi}_i = \boldsymbol{\Phi}_{i,G}$

  end

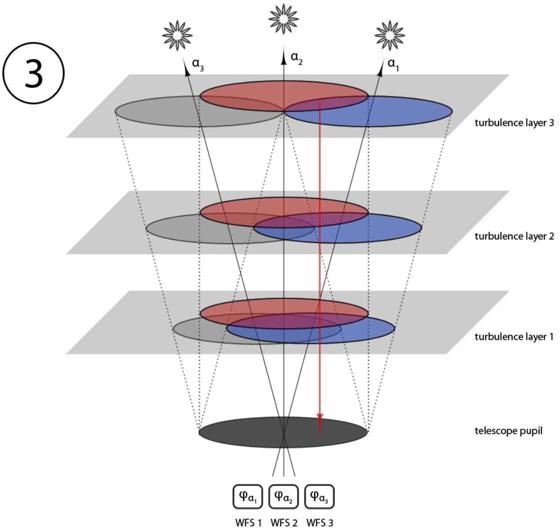
The Iteration



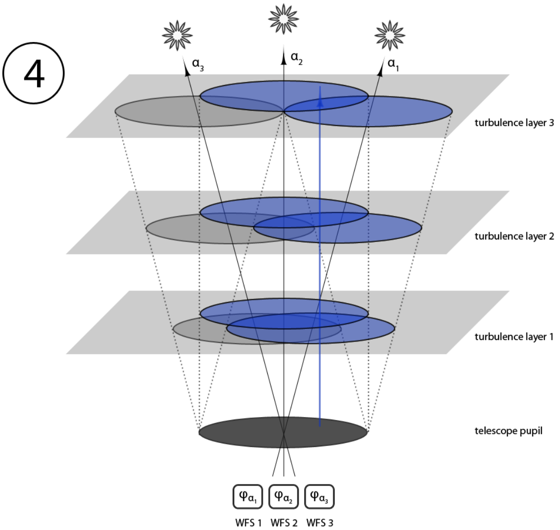
The Iteration



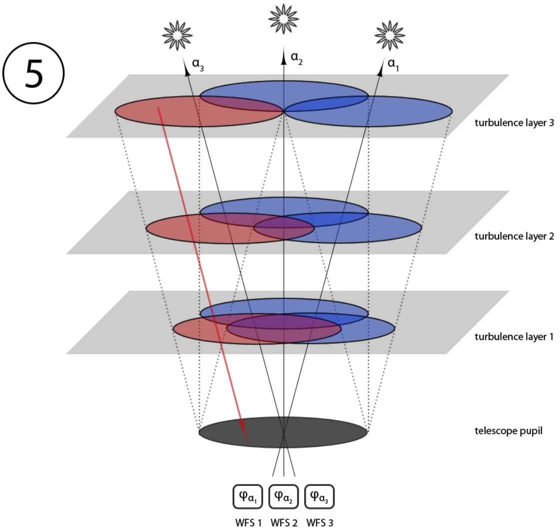
The Iteration



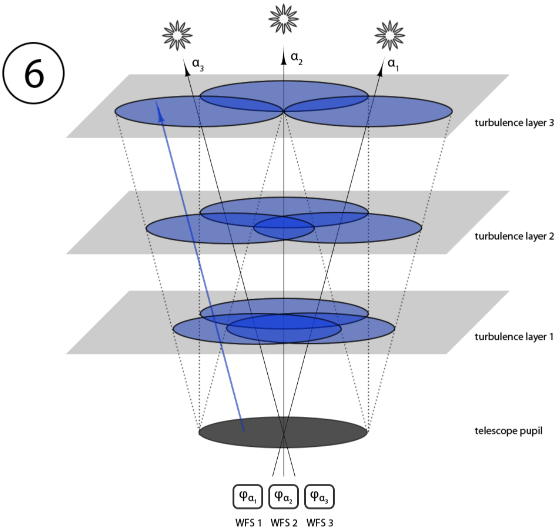
The Iteration



The Iteration



The Iteration



Kaczmarz iteration - analytical properties for atmospheric tomography

1. iteration can be evaluated without using matrix - vector operations
   $\Longrightarrow$ each iteration numerically cheap!
2. convergences geometrically to a least squares solution
3. regularization by early termination of the iteration
4. closed loop: few iterations are sufficient!

The fitting step

Need to obtain the shape of the deformable mirror

- M deformable mirrors, conjugated to altitudes $\tilde{h}_m, m = 1, \ldots, M$
- shape of the mirrors is described by

$$\tilde{\mathbf{\Phi}}_{DM} = \left(\tilde{\Phi}_{DM}^{(1)}, \ldots, \tilde{\Phi}_{DM}^{(M)} \right)$$

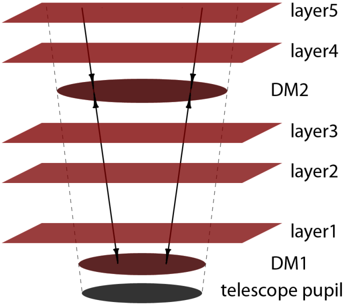
- Input: reconstructed atmosphere

$$\tilde{\mathbf{\Phi}} = \left(\tilde{\Phi}^{(1)}, \ldots, \tilde{\Phi}^{(L)} \right),$$

- optimize w.r.t. set of D directions (e.g., 25 evaluation directions)

$$\tilde{\boldsymbol{\alpha}} = \{ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_D \}$$

Optimal mirror fitting



$$\sum_{d=1}^{D} \| \sum_{m=1}^{M} \tilde{\Phi}_{DM}^{(m)}(\cdot + \tilde{\alpha}_d \tilde{h}_m) - \sum_{l=1}^{L} \tilde{\Phi}^{(l)}(\cdot + \tilde{\alpha}_d \tilde{h}_l) \|^2 \longrightarrow min!$$

- functional can be optimized with few Kaczmarz iterations
- optimization directions = guide star directions $\sim$ reconstruction of artificial layers at DM location

Test case setting

Telescope setting:

- telescope diameter: 42m
- Shack-Hartmann WFSs (84×84)
- 6 LGSs (circle with diam 2 arcmin)
- 3 NGSs (circle with diam 2.66 arcmin)
- bilinear ansatz functions for representation of the layers

Atmospheric setting:

- $r_0 = 12.9$cm
- 9 layers assumed in the atmosphere
- 500 Hz
- AO simulation of 1 seconds

Quality tests on OCTOPUS

- temporal control with pseudo open-loop control
- tests for two different approaches
  - 3-layer reconstruction on DM heights
    - atmospheric reconstruction and fitting in one step
    - no knowledge on the atmosphere needed
    - very low computational effort
  - 9-layer reconstruction with optimal fitting
    - incorporate knowledge on the atmosphere
    - can select optimization directions
    - higher quality

Computational complexity and speed

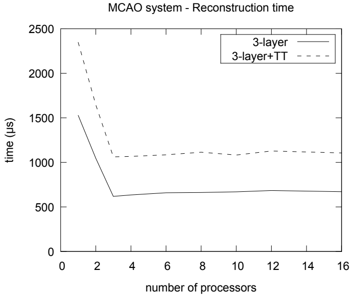
Complexity for each iteration (SH-WFS 84×84, $n = 0.75 * 84^2 = 5292$):

- $(18 * \#L - 14) * \#gs * n$
- 3 layers, 6 guide stars : 240n
- 9 layers, 6 guide stars: 888n

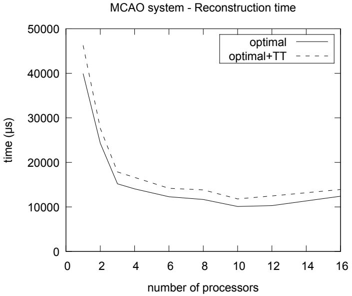
additionally $20 * n$ per guide star for reconstruction of wavefront (CuReD)

- the projection operator can be parallelized

3-layer reconstruction (μs)

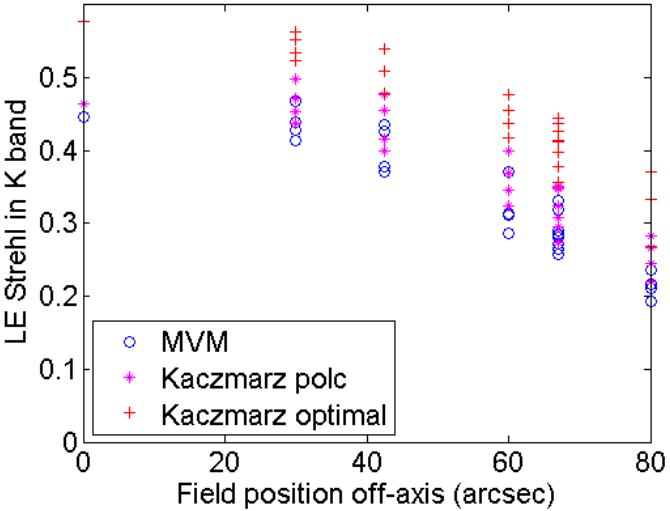


9-layer reconst., optimal fitting (μs)

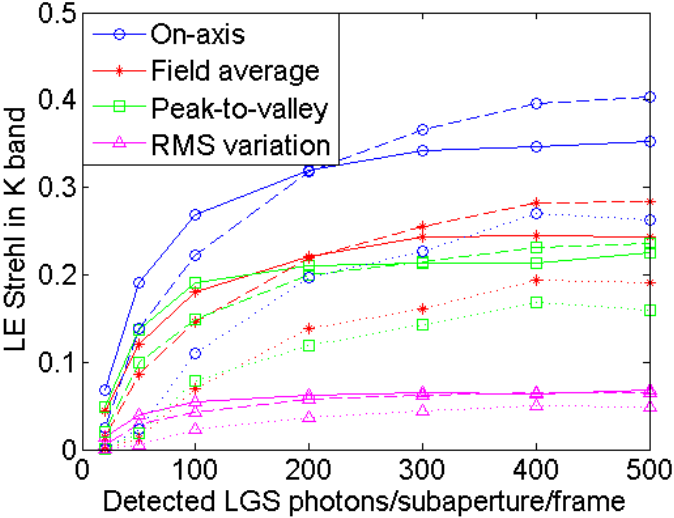


- MVM time: $130ms$

Reference case - High flux, no elongation



Reference case - Strehl vs. flux, with spot elongation



(solid . . . MVM, dashed . . . Kaczmarz optimal, dotted . . . 3-layer Kaczmarz)