Dynamics and Stability of Triple Stars

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Summary. The dynamics of triple stars and methods for computing the motions are briefly reviewed: The topics include the statistical properties of unstable triples, stability limits in hierarchical systems and numerical methods to compute the evolution and the largest Lyapunov exponent of a triple star model.

1 Introduction

Evolution of (point-mass) three-body systems is qualitatively similar in future and past: the system disrupts (*unstable*) ejecting one of the stars, in both direction of time, or it stays bounded (*stable*) forever.

Consequently any encounter of a binary and a single star, however complicated, eventually leads to ejection of one of the stars leaving behind a binary.

Known stable triple systems are hierarchical. An interesting case is high mutual inclination with strong eccentricity variation (Kozai-resonance). Other cases include: the 'figure-8' (quasi-)periodic system, some nearly rectilinear orbits and co-orbital solutions.

The astrophysically most interesting cases are:

1. Systems of three stars that break up after some dynamical evolution. These may be found in star forming regions.
2. Scattering of single stars off close binaries. These events are important in the dynamics of star clusters.
3. Stable hierarchical systems. Numerous such triple stars are known in the galactic field.

Discussions of the astrophysical applications of the three-body dynamics has been provided e.g. by Valtonen and collaborators [32, 33, 34].

2 Disruption of Triple Systems and Scattering off Binaries

2.1 Cross-Sections and Thermal Distributions

An important phenomenon in stellar dynamics is the scattering of single stars off binaries which has been extensively studied by Hut and collaborators [13, 14, 15, 16, 10, 11, 19, 12].

In that process energy flows from the binaries to single stars thus heating the stellar system. In dense systems this prevents the (total) core collapse.

¿From the numerical experiments an approximate summary for the relative energy exchange (Δ) cross-section σ can be expressed as

$$\frac{d\sigma}{d\Delta} \approx 2\pi A \left(\frac{V_c}{V}\right)^2 \Delta^{-0.5}(1+\Delta)^{-4}, \tag{1}$$

where the coefficient $A \approx 21$ for equal masses, V is the incoming speed of the third star while V_c is the critical value at which the total energy is zero.

The first comprehensive treatment of three-body scattering was given by Heggie [9]. An important result was that hard binaries get harder and soft ones typically disrupt. Here the boundary between a hard and a soft can be defined in terms of the binding energy of the binary: if the binding energy is larger than a typical kinetic energy of a single star, the binary is hard. Otherwise soft.

As a result of the scattering process the final binding energy (B) distribution of binaries is expected to be [9, 35]

$$f(B) \propto B^{-4.5}, \tag{2}$$

while the square of eccentricity is usually nearly uniformly distributed. For the eccentricity this can be written

$$f(e) = 2e. \tag{3}$$

High eccentricities are thus expected to be common in binaries that have experienced three-body encounters.

These results are valid in the ‘thermal equilibrium’ in which each event is balanced by an equally probable inverse one [9]. Valtonen and Karttunen [36] recently used phase space volume arguments to arrive at similar results.

2.2 Probability of escape

The probability of escape of a given star from a strongly interacting triple system depends on the mass

$$P_{esc}(m_k) \approx m_k^{-n} / \sum_\nu m_\nu^{-n}, \tag{4}$$

with a value of $n \approx 2$ applicable in typical triple interactions. This result can be derived from phase-space volume-integrals (in agreement with numerical experiments), and it explains e.g. the fact that the mass ratio distribution varies with the spectral class [35].

However, if considered in more detail, the escape probability exponent n depends significantly on the angular momentum of the system. More precisely, one may write

$$n \approx 3/(1 + \lambda/3), \tag{5}$$

where λ is the dimensionless scale-independent parameter

$$\lambda = -c^2 E \langle m \rangle \langle m_i m_j \rangle^{-3} G^{-2}, \tag{6}$$

in which the angular brackets indicate the mean mass and the mean mass product respectively and G is the gravitational constant [22].

One notes that another form of the above equation is

$$P_{ij} = (m_i m_j)^n / \sum_{\alpha\beta} (m_\alpha m_\beta)^n, \tag{7}$$

where P_{ij} is the probability that the pair $m_i m_j$ is the surviving binary. This form has the advantage that one may apply it to more complicated systems. For 4-body systems in which a common outcome is one binary and two independent stars, the value $n \approx 2$ has been obtained in case of low angular momentum [24]. Again, smaller values for n may be expected for larger angular momentum.

2.3 Chaos or not?

The three-body motions typically change significantly if the the initial conditions are changed a little. This, however, does not always mean that the motions are actually chaotic, but that the phase space is very complicated and is divided into areas of different behavior. The (hyper) surfaces which divide the phase space are associated with orbits leading to a parabolic disruption of the system or a triple collision.

Figure 1 illustrates one aspect of the outcome of triple scattering: The impact parameter was changed in small intervals, the system was integrated to final disruption and the semi-major axis (over the maximum possible value) was plotted in the figure. One notes regions of regular behavior (U-shaped portions of the curve) as well as regions of high sensitivity. However, those chaotic looking sections are actually filled with very narrow U-shaped curves. At the points where $a/a_{max} = 1$ the third star escapes with (asymptotically) parabolic speed (=0 at infinity). Such an orbit is, however, infinitely sensitive to initial conditions because, on one side, there is a hyperbolic escape and, on the other side, a long ejection without escape (yet). One may thus say that we see here dense systems of singular surfaces of parabolic disruption

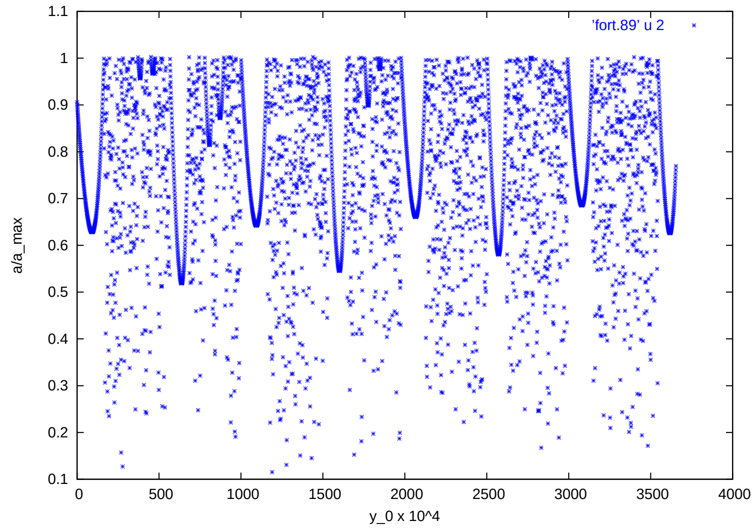

Fig. 1. $z = a/a_{max}$ as function of impact parameter ($\times 10^4$). The value $z = 1$ corresponds to parabolic disruption.

[25]. However, near the parabolic escape orbit, in the side where there is no disruption but long ejections, there is a fractal-like structure, as shown by Boyd and McMillan [4].

3 Long Lasting Stability

As far as is known, all observed stable triple stars are hierarchical systems, although theoretically other types of stable systems exist. In this section, first the stability conditions for hierarchical triples are discussed and then some more "exotic" systems are considered.

3.1 Hierarchical Triples

The stability of hierarchical triple stars is largely determined by the pericenter distance of the outer orbit. There are many studies of this in the literature e.g. by Harrington [7], Bailyn [3], Kiseleva and Eggleton [6], Mardling and Aarseth [21, 1]. Those authors give various estimates for the ratio of the pericentre distance R_{peri} of the outer orbit to the semi-major axis a_{in} of the inner orbit:

$$\left(\frac{R_{peri}}{a_{in}}\right)_{\text{Harrington}} = 3.5\left[1 + 0.7\ln\left(\frac{2}{3} + \frac{2}{3}\frac{m_3}{m_1 + m_2}\right)\right] \qquad (8)$$

$$\left(\frac{R_{peri}}{a_{in}}\right)_{\text{Bailyn}} = \frac{2.65 + e_{in}}{3.5}\left(1 + \frac{m_3}{m_1 + m_2}\right)^{\frac{1}{3}}\left(\frac{R_{peri}}{a_{in}}\right)_{\text{Harrington}} \quad (9)$$

$$\left(\frac{R_{peri}}{a_{in}}\right)_{\text{Egg-Kise}} = (1 + e_{in})\left(1 + \frac{3.7}{Q_3} - \frac{2.2}{1 + Q_3} + \frac{1.4}{Q_2}\,\frac{Q_3 - 1}{Q_3 + 1}\right) \quad (10)$$

$$\left(\frac{R_{peri}}{a_{in}}\right)_{\text{MardlingAarseth}} = 2.8\left[\left(1 + \frac{m_3}{m_1 + m_2}\right)\frac{1 + e_{out}}{\sqrt{1 - e_{out}}}\right]^{\frac{2}{5}}. \quad (11)$$

Here $Q_2 = [max(m1/m2, m2/m1)]^{\frac{1}{3}}$, $Q_3 = (\frac{m_1+m_2}{m_3})^{\frac{1}{3}}$ in the Eggleton-Kiseleva criterion. The masses m_1, m_2 are the components of the inner binary and m_3 is the outer body and the indices *in* and *out* refer to the inner and outer orbits. The Eggleton-Kiseleva criterion is for circular orbits only (i.e. both inner and outer are assumed circles initially) while the other expressions are supposed to be useful more generally. However, as we can see, these expressions depend in a different way on masses and orbital elements. They typically give values for $(\frac{R_{peri}}{a_{in}})$ in the range from 3 to 4, although larger values occur in extreme cases. By numerical experiments it is not difficult to find examples in contradiction with the values obtained from these estimates. Thus none of the published estimates is sufficiently accurate for deeming the stability of a triple system with certainty. In practice, especially for cases in the mentioned range, it is better to determine the stability by numerical integration.

Figure 2 illustrates the motion of a stable triple (but near instability boundary). In Fig. 3 the variational equation solutions for this triple and for a close unstable one are plotted. One can see that the instability becomes quickly evident from the behavior of the variation.

3.2 Kozai resonance

If the mutual inclination in a triple system is high enough, the system undergoes strong periodic variations in the inner eccentricity and the mutual inclination, known as the Kozai resonance [18, 17]. Especially if the orbits are perpendicular, the inner eccentricity should reach the value $e = 1$! This may thus restrict the existing triples to those having small mutual orbital inclinations (or near 180 degrees). On the other hand, the Kozai resonance is sensitively affected by other effects, such as oblateness of the bodies. Otherwise the existence of the moons of Uranus, or the triple system Algol, would not be possible.

3.3 Non-hierarchical Triples

Recently Montgomery [20] discovered an interesting special stable triple orbit in which all the three bodies move along an figure-8 trajectory with one third phase difference. This orbit is illustrated in Fig. 4. Since the periodic orbit

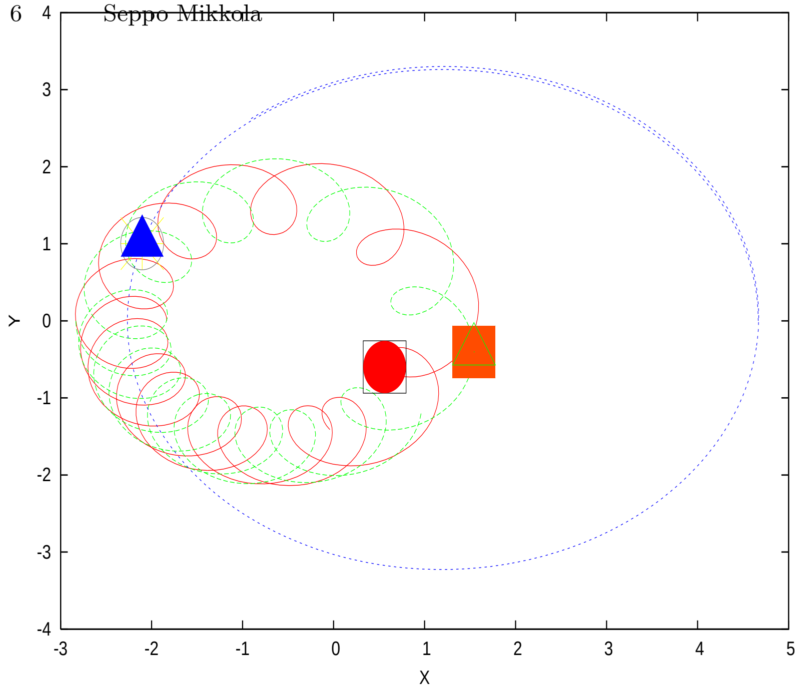

Fig. 2. Motion of a hierarchical stable triple in centre-of-mass coordinates. The big dots describe the positions of the particles at a selected moment.

is stable, one can expect stable motions in the neighborhood of the basic solution.

Other such cases can be found in the neighborhood of other stable periodic orbits such as the Broucke orbit [5], the rectilinear Schubart orbit [31, 26], the co-orbital Lagrangian solutions and the Copenhagen problem. In some cases some of the masses of the bodies must be small (Lagrangian and Copenhagen problems). These special orbits, although very interesting for a theoretician, are less important in astrophysics.

4 Numerical Methods for Triple Stars

Classical numerical methods can be used to compute the motions of triple stars if close approaches do not occur. Typically, however, regularized methods are more accurate even when the interactions are not particularly strong. Today there are several alternative methods available that utilize the Kustaanheimo-Stiefel (KS) transformation. These include the method of Aarseth and Zare [2], the global regularization of Heggie [8, 23], and more recent ones like the logarithmic Hamiltonian method [27, 30]. Details can be found e.g. in the book by Aarseth [1].

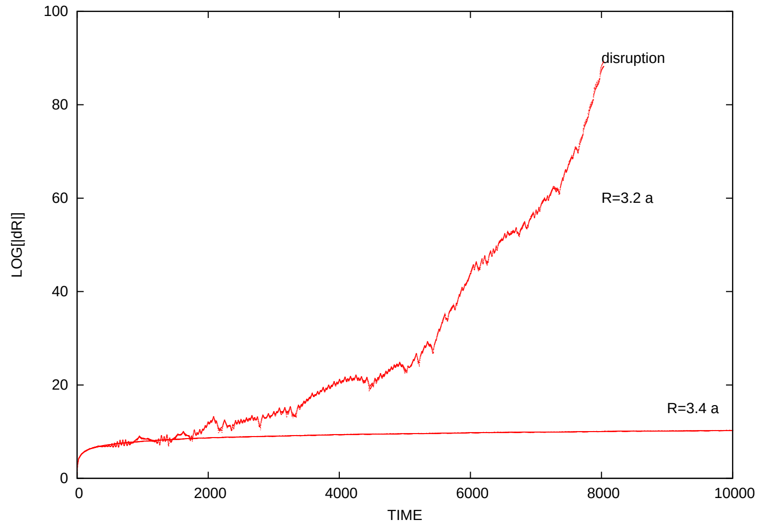

Fig. 3. Evolution of $\ln(|dR|)$ for two hierarchical three-body orbits $[R_{out} = 3.2a_{in}$ and $R_{out} = 3.4a_{in}]$. Masses=1, $a = 5$.

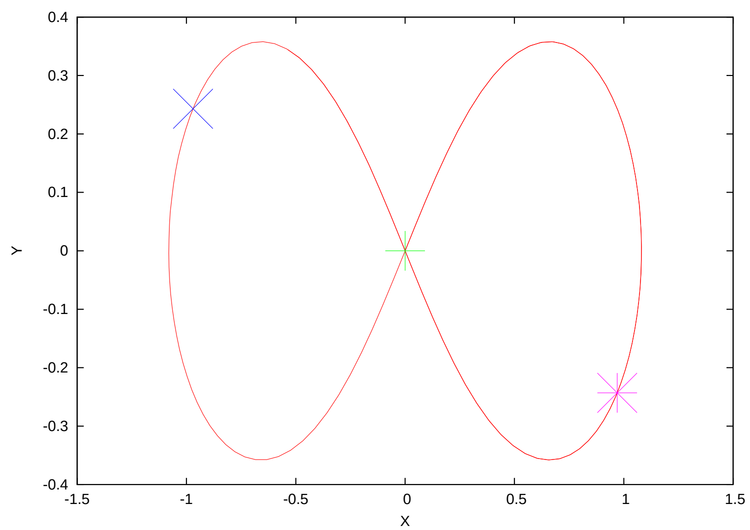

Fig. 4. The figure-8 (stable, periodic) solution.

4.1 A Regular Symplectic Three-Body Algorithm

Here we consider in more detail only one method which has the advantage that it is simple enough to allow straightforward differentiation of the algorithm thus making it possible to obtain easily the largest Liapunov exponent (i.e. determination of system stability). This method is based on the logarithmic Hamiltonian formalism [27, 30].

Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ be the position vectors of the bodies m_1, m_2, and m_3 and let us introduce the difference vectors

$$\mathbf{d}_1 = \mathbf{r}_3 - \mathbf{r}_2,\ \mathbf{d}_2 = \mathbf{r}_1 - \mathbf{r}_3,\ \mathbf{d}_3 = \mathbf{r}_2 - \mathbf{r}_1. \tag{12}$$

Then the equations of motion can be written (in units in which the gravitational constant is one)

$$\dot{\mathbf{w}}_k = \mathbf{A}_k(\mathbf{d}) = -M\frac{\mathbf{d}_k}{d_k^3} + m_k \sum_{j=1}^{3} \frac{\mathbf{d}_j}{d_j^3} \tag{13}$$

$$\dot{\mathbf{d}}_k = \mathbf{w}_k, \tag{14}$$

where thus $\mathbf{w}$'s are the derivatives of the relative vectors $\mathbf{d}_k$ and $M = m_1 + m_2 + m_3$ is the total mass. The kinetic energy (in the centre-of-mass system) is

$$T = \frac{1}{2M}(m_1 m_2 \mathbf{w}_3^2 + m_1 m_3 \mathbf{w}_2^2 + m_2 m_3 \mathbf{w}_1^2) \tag{15}$$

and the potential energy

$$U = \frac{m_1 m_2}{d_3} + \frac{m_1 m_3}{d_2} + \frac{m_2 m_3}{d_3}. \tag{16}$$

Using the constant total energy $E = T - U$, which is evaluated only once at the beginning, one may now write new time-transformed equations of motion (which now include also an equation for the time t)

$$\mathbf{w}' = \mathbf{A}(\mathbf{d})/U \tag{17}$$

$$\mathbf{d}' = \mathbf{w}/(T - E) \tag{18}$$

$$t' = 1/(T - E), \tag{19}$$

where the equality $U = T - E$ is used to make the derivatives of $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ depend only on the coordinates $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ and the derivatives of $\mathbf{d}$ to depend only on $\mathbf{w}$.

Thus the simple leapfrog algorithm is possible. Since in this case the leapfrog is exact for two-body motion [27, 30], we have an algorithm that is regular in two-body collisions (algorithmic regularization) even if the differential equations are singular.

Defining the two 'subroutines' $\mathbf{X}(s)$ and $\mathbf{V}(s)$ (where s is a step size),

$$\mathbf{X}(s): \quad \delta t = s/[T(\mathbf{w}) - E]; \quad \mathbf{d} \to \mathbf{d} + \delta t\, \mathbf{w}; \quad t \to t + \delta t \tag{20}$$

$$\mathbf{V}(s): \quad \widetilde{\delta t} = s/U(\mathbf{d}); \quad \mathbf{w} \to \mathbf{w} + \widetilde{\delta t}\, \mathbf{A}(\mathbf{d}), \tag{21}$$

one can symbolize one step of the leapfrog algorithm as

$$\mathbf{X}(h/2)\mathbf{V}(h)\mathbf{X}(h/2)$$

or, if several steps are taken between outputs,

$$\mathbf{X}(h/2)\mathbf{V}(h)\mathbf{X}(h)\mathbf{V}(h).....\mathbf{X}(h)\mathbf{V}(h)\mathbf{X}(h/2)$$

i.e. the half-steps are taken only in the beginning and at the end (output).

This algorithm is simple to program, not singular in collision and can be used also for a soft potential model in which $1/r$ is replaced by $1/\sqrt{r^2+\epsilon^2}$. The method is also symplectic and an improvement of accuracy over the Yoshida's higher-order leapfrogs [37] or the extrapolation method [29] is possible.

Due to the structure of the leapfrog, this method exactly conserves the angular momentum, as well as the geometric integrals $\sum_k \mathbf{d}_k = 0$ and $\sum_k \mathbf{w}_k = 0$. One could also integrate only two of the relative vectors $\mathbf{d}_k$ and obtain the third one from the geometric integrals. However, in practice this hardly saves any computational effort. Instead, one may occasionally use the geometric integrals to remove any round-off effect by computing the largest side from the sum of the two others, and applying the same for the corresponding velocities.

An additional important feature of this algorithm is that it is very easy to differentiate so as to obtain the tangent map [28] and thus the maximum Lyapunov exponent. What one does in practice is that the code is first written and then differentiated line by line. Thus one obtains the exact differentials of the algorithm, essentially without considering the variational equations.

5 Conclusion

The scattering of single stars off binaries and the disruption of triple systems are rather well understood, especially in the statistical sense.

However, the stability of hierarchical triple stars still lacks a reliable theoretical estimate. Thus, to determine the stability properties of any triple system model, a check by numerical integration, preferably with a computation of the largest Lyapunov exponent, is recommended.

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