Galaxy Formation and Fluctuations in the Cosmic Microwave Background Radiation

The Basic Physics of What You Really Need to Know

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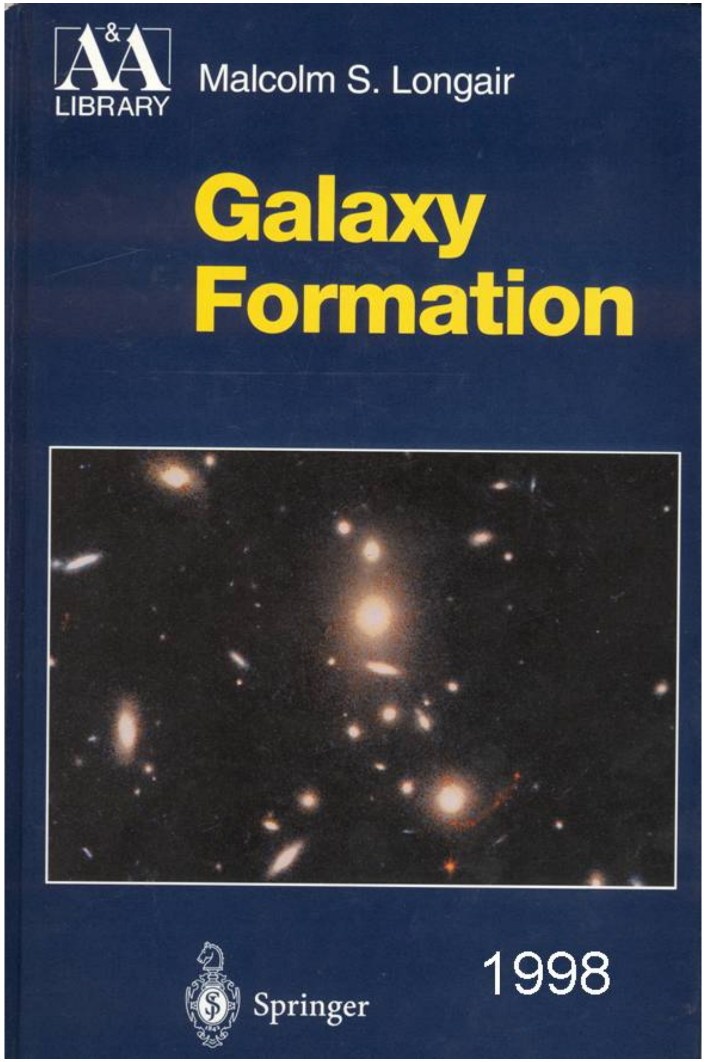
- The Physics of the Friedman equations
- The Relevant Solutions
- The Basic Physics of Galaxy and Structure Formation
- How Dark Matter Saves the Day
- The Results of Wilkinson Microwave Anisotropy Probe and the Sloan Digital Sky Survey

Book of the Lecture

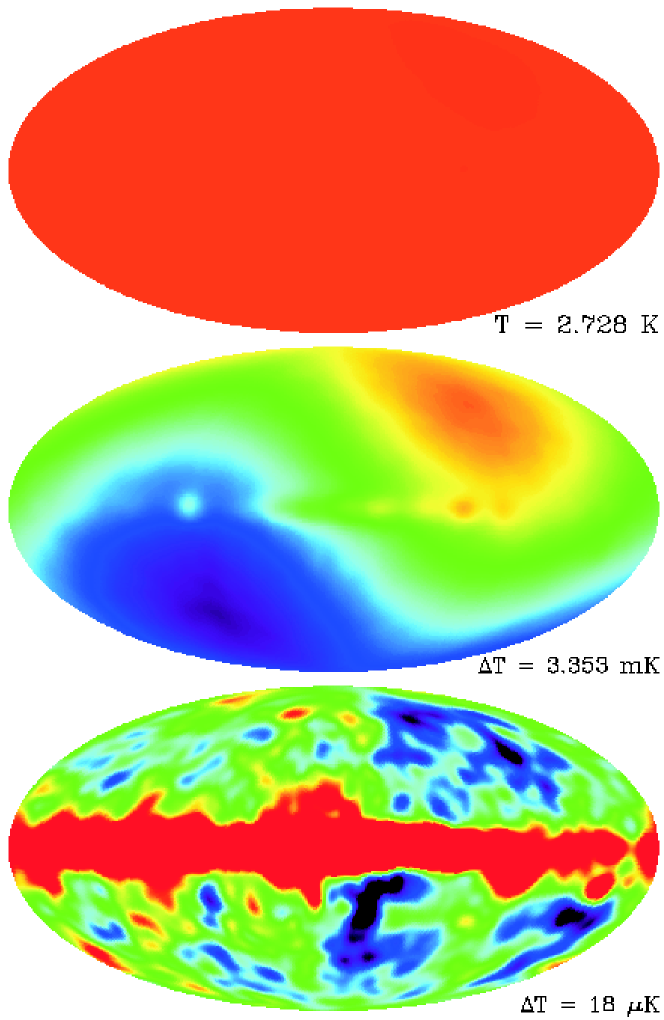


I will be using the material from this book but bringing the story up-to-date.

The emphasis will be upon understanding the basic physics involved in the standard concordance picture. I will try to keep the physics as simple as possible.

I am rewriting this book at the moment - suggestions for material to be included will be welcomed.

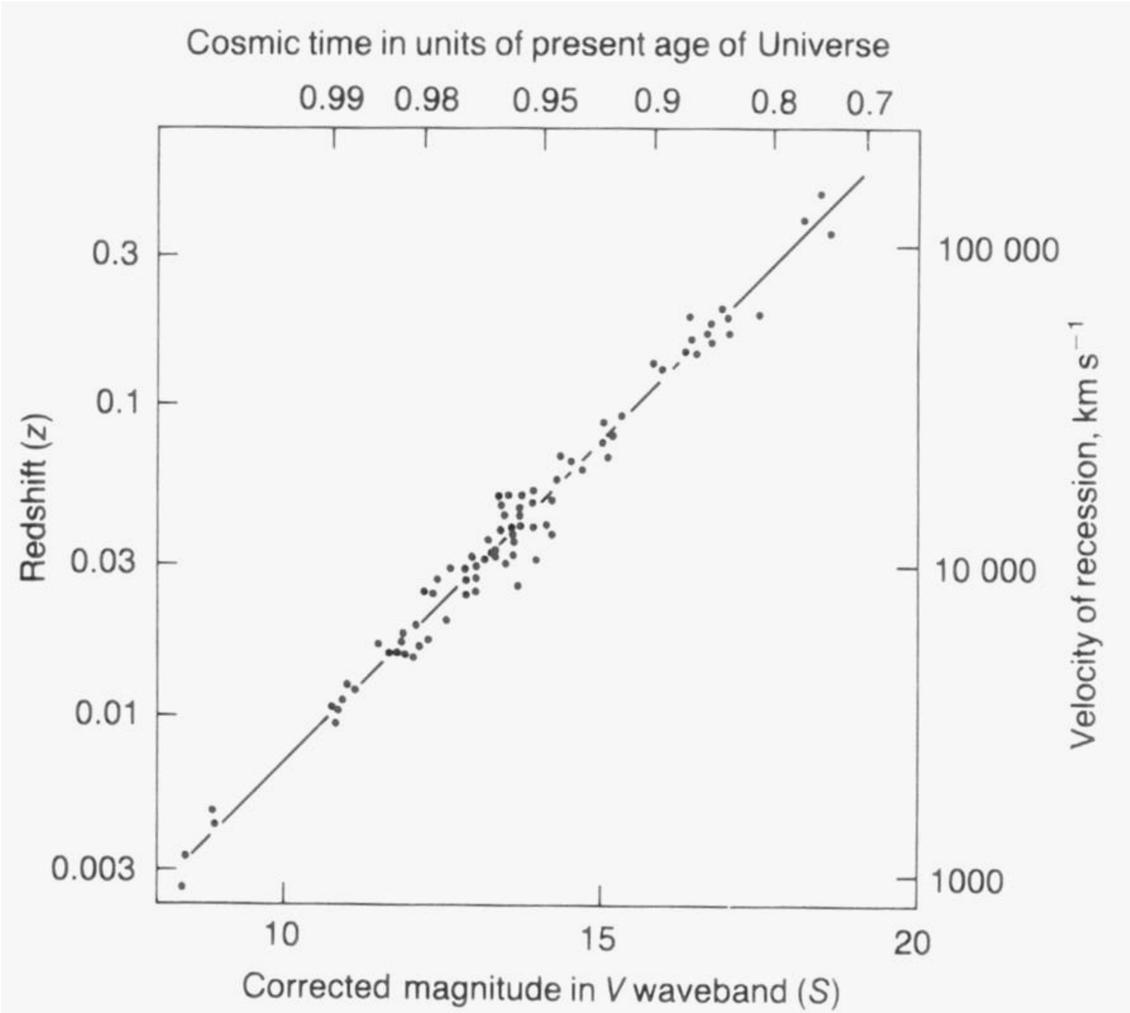
COBE Observations of the Cosmic Microwave Background Radiation (1990s)



The starting points for cosmological studies nowadays are the observations of the Cosmic Microwave Background Radiation by the COBE satellite in the early 1990s.

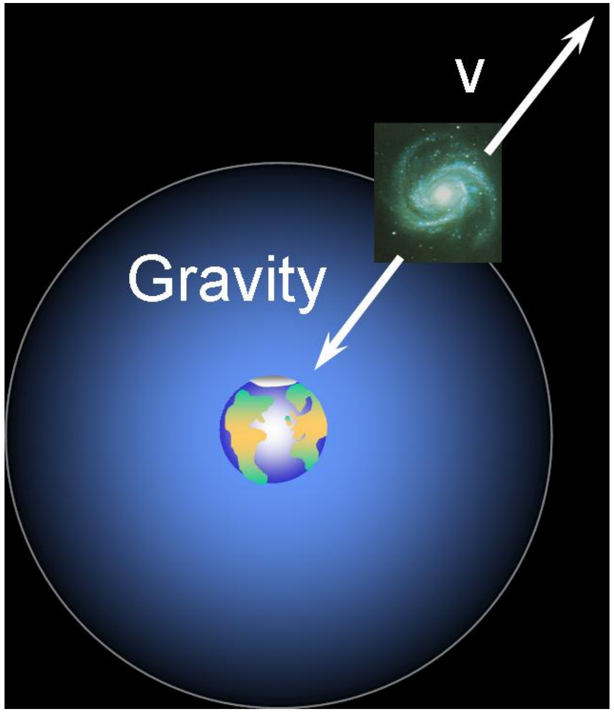
- The spectrum is very precisely that of a perfect black-body at a radiation temperature of 2.726 K.
- A perfect dipole component is detected, corresponding to the motion of the Earth through the frame in which the radiation would be perfectly isotropic.
- Away from the Galactic plane, the radiation is isotropic to better than one part in 10^5. At this level, significant temperature fluctuations $\Delta T/T \approx 10^{-5}$ were detected on scales $\theta \geq 10^\circ$.

Hubble's Law



A modern version of Hubble's law for the brightest galaxies in rich clusters of galaxies, $v = H_0 r$. All classes of galaxy seem to follow the same Hubbles law. H_0 is Hubbles constant.

Newtonian Cosmological Models



In 1934, Milne and McCrea showed that the structure of the Friedman equations can be derived using non-relativistic Newtonian dynamics. Consider a galaxy at distance x from the Earth and determine its deceleration due to the gravitational attraction of the matter inside the sphere of radius x centred on the Earth. By Gauss's theorem, because of the spherical symmetry of the distribution of matter within x, we can replace that mass, $M = (4\pi/3)\varrho x^3$, by a point mass at the centre of the sphere and so the deceleration of the galaxy is

$$m\ddot{x} = -\frac{GMm}{x^2} = -\frac{4\pi x \varrho m}{3}\,. \qquad (1)$$

The mass of the galaxy m cancels out on either side of the equation, showing that the deceleration refers to the sphere of matter as a whole rather than to any particular galaxy.

We now introduce comoving coordinates. We are dealing with isotropic Universes which expand uniformly. We therefore introduce the concept of comoving distance. If the distance between two points expanding with the Universe is R and R_0 is their separation at the present epoch, we can write $x = (R/R_0)r$ and so take out the expansion of the Universe. I will normally set the scale factor equal to unity at the present epoch, $R_0 = 1$ for simplicity. R is the scale factor.

We can also express the density in terms of its value at the present epoch, $\varrho = \varrho_0 R^{-3}$. Therefore,

$$\boxed{\ddot{R} = -\frac{4\pi G \varrho_0}{3R^2}} \quad \text{or} \quad \boxed{\ddot{R} = -\frac{4\pi G \varrho R}{3},} \tag{2}$$

Multiplying (3) by $\dot{R}$ and integrating, we find

$$\boxed{\dot{R}^2 = \frac{8\pi G \varrho_0}{3R} + \text{constant}} \quad \text{or} \quad \boxed{\dot{R}^2 = \frac{8\pi G \varrho R^2}{3} + \text{constant.}} \tag{3}$$

This Newtonian calculation shows that we can identify the left-hand side of (3) with the kinetic energy of expansion of the fluid and the first term on the right-hand side with its gravitational potential energy.

Einstein's Field Equations

In the full GR analysis, Einstein's field equations reduce to the following pair of independent equations.

$$\ddot{R} = -\frac{4\pi G}{3} R \left(\varrho + \frac{3p}{c^2} \right) + \left[\tfrac{1}{3} \Lambda R \right] \; ; \tag{4}$$

$$\dot{R}^2 = \frac{8\pi G \varrho}{3} R^2 - \frac{c^2}{\Re^2} + \left[\tfrac{1}{3} \Lambda R^2 \right] \; . \tag{5}$$

In these equations, R is the scale factor, ϱ is the total inertial mass density of the matter and radiation content of the Universe and p the associated total pressure. $\Re$ is the radius of curvature of the geometry of the world model at the present epoch and so the term $-c^2/\Re^2$ is simply a constant of integration. The *cosmological constant* Λ, which has been included in the terms in square brackets in (4) and (5), has had a chequered history since it was introduced by Einstein in 1917.

The Meaning of the Term $\varrho + \frac{3p}{c^2}$

Let us look more closely at the meanings of the various terms. Equation (5) is referred to as *Friedman's equation* and has the form of an energy equation. The First Law of Thermodynamics in its relativistic form needs to be built into this equation. We can write it in the usual form

$$\mathrm{d}U = -p\,\mathrm{d}V\,. \tag{6}$$

We need to formulate the first law in such a way that it is applicable for relativistic and non-relativistic fluids and so we write the internal energy U as the sum of all the terms which can contribute to the total energy of the fluid in the relativistic sense. Thus, the total internal energy consists of the fluid's rest mass energy, its kinetic energy, its thermal energy and so on. If we write the sum of these energies as $\varepsilon_{\mathrm{tot}} = \sum_i \varepsilon_i$, the internal energy is $\varepsilon_{\mathrm{tot}} V$ and so, differentiating (6) with respect to R, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}R}(\varepsilon_{\mathrm{tot}} V) = -p\,\frac{\mathrm{d}V}{\mathrm{d}R}\,. \tag{7}$$

Now, $V \propto R^3$ and so, differentiating, we find

$$\frac{\mathrm{d}\varepsilon_{\mathrm{tot}}}{\mathrm{d}R} + 3\frac{(\varepsilon_{\mathrm{tot}} + p)}{R} = 0\,. \tag{8}$$

This result can be expressed in terms of the inertial mass density associated with the total energy $\varepsilon_{\mathrm{tot}} = \varrho c^2$ and so (8) can also be written

$$\frac{\mathrm{d}\varrho}{\mathrm{d}R} + 3\frac{\left(\varrho + \frac{p}{c^2}\right)}{R} = 0\,. \tag{9}$$

This is the type of density ϱ which should be included in (4) and (5).

In the case of a gas of *ultrarelativistic particles*, or a *gas of photons*, we can write $p = \frac{1}{3}\varepsilon_{\mathrm{tot}}$. Therefore,

$$\frac{\mathrm{d}\varepsilon_{\mathrm{tot}}}{\mathrm{d}R} + \frac{4\varepsilon_{\mathrm{tot}}}{R} = 0 \quad \text{and so} \quad \varepsilon_{\mathrm{tot}} \propto R^{-4}\,. \tag{10}$$

In the case of a gas of photons, $\varepsilon_{\mathrm{rad}} = \sum N h\nu$ and, since $N \propto R^{-3}$, we find $\nu \propto R^{-1}$. This is just the formula for redshift.

Let us now return to the analysis of (5). Differentiating

$$\dot{R}^2 = \frac{8\pi G\varrho}{3}R^2 - \frac{c^2}{\Re^2} + \left[\tfrac{1}{3}\Lambda R^2\right] \; . \tag{11}$$

with respect to time and dividing through by $\dot{R}$, we find

$$\ddot{R} = \frac{4\pi G R^2}{3}\frac{\mathrm{d}\varrho}{\mathrm{d}R} + \frac{8\pi G\varrho R^2}{3} + \left[\tfrac{1}{3}\Lambda R\right] \; . \tag{12}$$

Now, substituting the expression for $\mathrm{d}\varrho/\mathrm{d}R$ from (9), we find

$$\boxed{\ddot{R} = -\frac{4\pi G}{3}R\left(\varrho + \frac{3p}{c^2}\right) + \left[\tfrac{1}{3}\Lambda R\right] \; ,} \tag{13}$$

that is, we recover (4).

Thus, equation (13) has the form of a force equation, but, as we have shown, it also incorporates the relativistic form of the First Law of Thermodynamics as well. This pressure term can be considered a ‘relativistic correction’ to the inertial mass density, but it is unlike normal pressure forces which depend upon the gradient of the pressure and, for example, hold up the stars. The term $\varrho + (3p/c^2)$ can be thought of as playing the role of an *active gravitational mass density*.

The Cosmological Constant Λ

In 1917, Einstein introduced the Λ-term in order to incorporate *Mach's principle* into General Relativity - namely that the local inertial frame of reference should be defined relative to the distant stars. In the process, he derived the first fully self-consistent cosmological model - the static Einstein model of the Universe.

Equation (4) is

$$\ddot{R} = -\frac{4\pi G}{3} R \left(\varrho + \frac{3p}{c^2} \right) + \left[\frac{1}{3} \Lambda R \right] . \tag{14}$$

Einstein's model is static and so $\ddot{R} = 0$ and the model is a 'dust model' in which the pressure is taken to be zero. Therefore,

$$\frac{4\pi G}{3} R \varrho = \frac{1}{3} \Lambda R \quad \text{or} \quad \boxed{\Lambda = 4\pi G \varrho} . \tag{15}$$

Einstein's perspective was that this formula shows that there would be no solutions of his field equations unless the cosmological constant was finite. If Λ were zero, the Universe would be empty.

The Cosmological Constant Λ

Let us consider the first of the field equations with finite Λ.

$$\ddot{R} = -\frac{4\pi G}{3} R \left(\varrho + \frac{3p}{c^2} \right) + \frac{1}{3} \Lambda R \, . \tag{16}$$

Even in an empty universe, with $\varrho = 0, p = 0$, there is a net force acting on a test particle. There is no obvious interpretation of this term in term of classical physics. There is, however, a natural interpretation in the context of quantum field theory.

A key development has been the introduction of *Higgs* fields into the theory of weak interactions. These were introduced in order to eliminate singularities in the theory and to endow the $W^{\pm}$ and Z^0 bosons with masses. Precise measurement of the masses of these particles at CERN has confirmed the theory very precisely. The Higgs fields are *scalar fields*, unlike the vector fields of electromagnetism or the tensor fields of General Relativity. The scalar fields have negative pressure equations of state $p = -\varrho c^2$.

The Cosmological Constant Λ

In the modern picture of the vacuum, there are zero-point fluctuations associated with the zero point energies of all quantum fields. The stress–energy tensor of a vacuum has a negative pressure equation of state, $p = -\varrho c^2$. This pressure may be thought of as a ‘tension’ rather than a pressure. When such a vacuum expands, the change in energy is $\mathrm{d}U = -p\,\mathrm{d}V$ in expanding from V to $V + \mathrm{d}V$ which is just $+\varrho c^2\,\mathrm{d}V$ so that, during the expansion, the mass-energy density of the negative energy field remains constant.

We can find the same result from (9).

$$\frac{\mathrm{d}\varrho}{\mathrm{d}R} + 3\frac{\left(\varrho + \frac{p}{c^2}\right)}{R} = 0\,.$$

It can be seen that, if the vacuum energy density is to remain constant, it follows that $p = -\varrho c^2$.

We can now relate ϱ_{V} to the value of Λ. We can now set $\Lambda = 0$ and instead include the energy and pressure of the vacuum fields into equation (16).

The Cosmological Constant Λ

$$\ddot{R} = -\frac{4\pi G R}{3}\left(\varrho_{\rm m} + \varrho_{\rm v} + \frac{3p_{\rm v}}{c^2}\right) , \tag{17}$$

where, in place of the Λ-term, we have included the density of ordinary mass $\varrho_{\rm m}$ and the mass density $\varrho_{\rm v}$ and pressure $p_{\rm v}$ of the vacuum fields. Since $p_{\rm v} = -\varrho_{\rm v} c^2$, it follows that

$$\ddot{R} = -\frac{4\pi G R}{3}(\varrho_{\rm m} - 2\varrho_{\rm v}) . \tag{18}$$

As the Universe expands, $\varrho_{\rm m} = \varrho_0 / R^3$ and $\varrho_{\rm v}$ = constant. Therefore,

$$\ddot{R} = -\frac{4\pi G \varrho_0}{3R^2} + \frac{8\pi G \varrho_{\rm v} R}{3} . \tag{19}$$

Equation (19) has precisely the same dependence upon R as of the 'cosmological term' and so we can formally identify the cosmological constant with the vacuum mass density.

$$\Lambda = 8\pi G \varrho_{\rm v} . \tag{20}$$

Density Parameters in the Matter and Vacuum Fields

Therefore, at the present epoch, $R = 1$, the first field equation becomes

$$\ddot{R}(t_0) = -\frac{4\pi G \varrho_0}{3} + \frac{8\pi G \varrho_v}{3} . \tag{21}$$

It is convenient to express densities in terms of the *critical density* ϱ_c defined by

$$\varrho_c = (3H_0^2/8\pi G) = 1.88 \times 10^{-26}\, h^2 \text{ kg m}^{-3} . \tag{22}$$

This is the density of the critical Einstein-de Sitter world model. Then, the actual density of the model ϱ_0 at the present epoch can be referred to this value through a *density parameter* $\Omega_0 = \varrho_0/\varrho_c$.

$$\Omega_0 = \frac{8\pi G \varrho_0}{3H_0^2} . \tag{23}$$

The subscript 0 has been attached to Ω because the critical density ϱ_c changes with cosmic epoch, as does Ω. It is convenient to refer any cosmic density to ϱ_c. For example, we will often refer to the density parameter of baryons, Ω_B, or of visible matter, Ω_{vis}, or of dark matter, Ω_{dark}, and so on – these are convenient ways of describing the relative importance of different contributions to Ω_0.

Density Parameter in the Vacuum Fields

A density parameter associated with ϱ_V can now be introduced, in exactly the same way as the density parameter Ω_0 was defined.

$$\Omega_\Lambda = \frac{8\pi G \varrho_\mathrm{V}}{3H_0^2} \quad \text{and so} \quad \Lambda = 3H_0^2\Omega_\Lambda \,. \tag{24}$$

The dynamical equations (4) and (5) can now be written

$$\ddot{R} = -\frac{\Omega_0 H_0^2}{2R^2} + \Omega_\Lambda H_0^2 R \,; \tag{25}$$

$$\dot{R}^2 = \frac{\Omega_0 H_0^2}{R} - \frac{c^2}{\Re^2} + \Omega_\Lambda H_0^2 R^2 \,. \tag{26}$$

A traditional way of rewriting these relations is in terms of a *deceleration parameter* q_0 defined by $q_0 = -\ddot{R}/\dot{R}^2$ at the present epoch. Then, in terms of Ω_0 and Ω_Λ, we find,

$$\boxed{q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda \,.} \tag{27}$$

Density Parameters in Matter and Vacuum Fields

We can now substitute the values of R and $\dot{R}$ at the present epoch, $R = 1$ and $\dot{R} = H_0$, into (26) to find the relation between the curvature of space, Ω_0 and Ω_Λ.

$$\frac{c^2}{\Re^2} = H_0^2[(\Omega_0 + \Omega_\Lambda) - 1] \; , \tag{28}$$

or

$$\kappa = \frac{1}{\Re^2} = \frac{[(\Omega_0 + \Omega_\Lambda) - 1]}{(c^2/H_0^2)} \; . \tag{29}$$

A common practice is to introduce a density parameter associated with the curvature of space at the present epoch Ω_K such that

$$\Omega_K = -\frac{c^2}{H_0^2 \Re^2} \tag{30}$$

Then, equation (29) becomes

$$\Omega_0 + \Omega_\Lambda + \Omega_K = 1 \; . \tag{31}$$

Density Parameters in Matter and Vacuum Fields

Thus, the condition that the spatial sections are flat Euclidean space becomes

$$(\Omega_0 + \Omega_\Lambda) = 1 \,. \tag{32}$$

The radius of curvature R_c of the spatial sections of these models change with scale factor as $R_c = R\Re$ and so, if the space curvature is zero now, it must have been zero at all times in the past. This is one of the great attractions of the simplest inflationary picture of the early Universe.

Estimating the Value of Ω_Λ

In their review of the problem of the cosmological constant, Carroll, Press and Turner described how a theoretical value of Ω_Λ could be estimated using simple concepts from quantum field theory. They found the mass density of the repulsive field to be $\varrho_{\rm V} = 10^{95}$ kg m^{-3}, about 10^{120} times greater than permissable values at the present epoch which correspond to $\rho_{\rm V} \leq 10^{-27}$ kg m^{-3}.

Heisenberg's Uncertainty Principle states that a virtual pair of particles of mass m can exist for a time $t \sim \hbar/mc^2$, corresponding to a maximum separation $x \sim \hbar/mc$. Hence, the typical density of the vacuum fields is $\rho \sim m/x^3 \approx c^3 m^4/\hbar^3$.

The mass density in the vacuum fields is unchanging with cosmic epoch and so, adopting the Planck mass for $m_{\rm Pl} = (hc/G)^{1/2} = 5.4 \times 18^{-8} = 3 \times 10^{19}$ GeV, for the mass associated with the quantum fluctuations in the gravitational field, the mass density corresponds to about 10^{97} kg m^{-3}. This is quite a problem. We have to explain why $\rho_{\rm V}$ decreased by a factor of about 10^{120} at the end of the inflationary era. In this context, 10^{-120} looks remarkably close to zero.

Key Results from the R-W Metric

- All the physics of the expansion of the Universe is built into the function $R(t)$, the scale factor. $R(t)$ is normalised to the value 1 at the present epoch $t = t_0$.
- The curvature of space $\Re$ changes with scale factor as $\Re(t) = \Re R$.
- By redshift, we mean the shift of spectral lines to longer wavelength because of their recession velocities from our Galaxy. If λ_e is the wavelength of the line as emitted and λ_0 the observed wavelength, the redshift z is defined to be

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} . \tag{33}$$

- It follows directly from the R-W metric that the redshift is directly related to the scale-factor R through the relation

$$R(t) = \frac{1}{1+z}. \tag{34}$$

This is the real meaning of redshift in cosmology.

The Concordance Model

This set of parameters is consistent with all observations listed above:

- Hubble's constant $H_0 = 72$ km s^{-1} Mpc^{-1}
- Baryonic density parameter $\Omega_B = 0.047$
- Cold Dark Matter density parameter $\Omega_D = 0.233$
- Total Matter density parameter $\Omega_0 = \Omega_B + \Omega_D = 0.28$
- Density Parameter in Vacuum Fields $\Omega_\Lambda = 0.72$
- Optical Depth for Thomson Scattering on Reheating $\tau = 0.17$
- Curvature of Space $\Omega_\Lambda + \Omega_0 = 1$; $\kappa = 0$.

For illustrative purposes, I will use these values in the calculations which follow.

The Properties of the Concordance Model

It therefore is sensible to regard this as the framework model for cosmological studies.

The Friedman equation is:

$$\dot{R}^2 = \frac{\Omega_0 H_0^2}{R} - \frac{c^2}{\Re^2} + \Omega_\Lambda H_0^2 R^2 \, . \tag{35}$$

Using the relation $R = 1/(1+z)$, we find

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -H_0(1+z)[(1+z)^2(\Omega_0 z + 1) - \Omega_\Lambda z(z+2)]^{1/2} \, . \tag{36}$$

Cosmic time t measured from the Big Bang follows immediately by integration

$$\boxed{t = \int_0^t \mathrm{d}t = -\frac{1}{H_0}\int_\infty^z \frac{\mathrm{d}z}{(1+z)[(1+z)^2(\Omega_0 z + 1) - \Omega_\Lambda z(z+2)]^{1/2}} \, .} \tag{37}$$

The Properties of the Concordance Model

The evidence suggests that we live in a Universe with zero spatial curvature, $\Re \to \infty$, and so $\Omega_0 + \Omega_\Lambda = 1$. This result simplifies the time-redshift relation:

$$t = \int_0^t \mathrm{d}t = -\frac{1}{H_0}\int_\infty^z \frac{\mathrm{d}z}{(1+z)[\Omega_0(1+z)^3 + \Omega_\Lambda]^{1/2}} \, . \tag{38}$$

The cosmic time–redshift relation becomes

$$t = \frac{2}{3H_0\Omega_\Lambda^{1/2}} \ln\left(\frac{1+\cos\theta}{\sin\theta}\right) \quad \text{where} \quad \tan\theta = \left(\frac{\Omega_0}{\Omega_\Lambda}\right)^{1/2}(1+z)^{3/2} \, . \tag{39}$$

The present age of the Universe follows by setting $z = 0$

$$t_0 = \frac{2}{3H_0\Omega_\Lambda^{1/2}} \ln\left[\frac{1+\Omega_\Lambda^{1/2}}{(1-\Omega_\Lambda)^{1/2}}\right] \, . \tag{40}$$

If we take $\Omega_\Lambda = 0.72$ and $\Omega_0 = 0.28$, the age of the world model is

$$T_0 = 0.983 H_0^{-1} = 1.32 \times 10^{10} \text{ years.} \tag{41}$$

Radiation Dominated Universes

The variations of p and ϱ with R can now be substituted into Einstein's field equations:

$$\ddot{R} = -\frac{4\pi G R}{3}\left(\varrho + \frac{3p}{c^2}\right) + \left[\tfrac{1}{3}\Lambda R\right] \; ;$$

$$\dot{R}^2 = \frac{8\pi G \varrho}{3} R^2 - \frac{c^2}{\Re^2} + \left[\tfrac{1}{3}\Lambda R^2\right] \; .$$

Therefore, setting the cosmological constant $\Lambda = 0$, we find

$$\ddot{R} = \frac{8\pi G \varepsilon_0}{3c^2}\frac{1}{R^3} \qquad \dot{R}^2 = \frac{8\pi G \varepsilon_0}{3c^2}\frac{1}{R^2} - \frac{c^2}{\Re^2} \, . \tag{42}$$

At early epochs we can neglect the constant term $c^2/\Re^2$ and integrating

$$R = \left(\frac{32\pi G \varepsilon_0}{3c^2}\right)^{1/4} t^{1/2} \quad \text{or} \quad \varepsilon = \varepsilon_0 R^{-4} = \left(\frac{3c^2}{32\pi G}\right) t^{-2} \, . \tag{43}$$

The dynamics of the radiation-dominated models, $R \propto t^{1/2}$, depend only upon the *total inertial mass density in relativistic and massless forms*. The force of gravity acting upon all the massless and relativistic components determines the rate of deceleration of the early Universe.

The Wave Equation for the Growth of Small Density Perturbations (1)

The standard equations of gas dynamics for a fluid in a gravitational field consist of three partial differential equations which describe (i) the conservation of mass, or the equation of continuity, (ii) the equation of motion for an element of the fluid, Euler's equation, and (iii) the equation for the gravitational potential, Poisson's equation.

$$\text{Equation of Continuity} \quad : \quad \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{v}) = 0 \; ; \tag{44}$$

$$\text{Equation of Motion} \quad : \quad \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\varrho}\nabla p - \nabla \phi \; ; \tag{45}$$

$$\text{Gravitational Potential} \quad : \quad \nabla^2 \phi = 4\pi G \varrho \, . \tag{46}$$

These equations describe the dynamics of a fluid of density ϱ and pressure p in which the velocity distribution is $\boldsymbol{v}$. The gravitational potential ϕ at any point is given by Poisson's equation in terms of the density distribution ϱ.

The partial derivatives describe the variations of these quantities *at a fixed point in space*. This coordinate system is often referred to as *Eulerian coordinates*.

Then, we perturb the system about the uniform expansion $\boldsymbol{v}_0 = H_0 \boldsymbol{r}$:

$$\boldsymbol{v} = \boldsymbol{v}_0 + \delta \boldsymbol{v}, \quad \varrho = \varrho_0 + \delta\varrho, \quad p = p_0 + \delta p, \quad \phi = \phi_0 + \delta\phi \,. \tag{47}$$

After a bit of algebra, we find the following equation for adiabatic density perturbations $\Delta = \delta\varrho/\varrho_0$:

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \frac{c_\mathrm{s}^2}{\varrho_0 R^2}\nabla_\mathrm{c}^2 \delta\varrho + 4\pi G \delta\varrho \,. \tag{48}$$

where the adiabatic sound speed c_s^2 is given by $\partial p/\partial\varrho = c_\mathrm{s}^2$. We now seek wave solutions for Δ of the form $\Delta \propto \exp \mathrm{i}(\boldsymbol{k}_\mathrm{c} \cdot \boldsymbol{r} - \omega t)$ and hence derive a wave equation for Δ.

$$\boxed{\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \Delta(4\pi G \varrho_0 - k^2 c_\mathrm{s}^2) \,,} \tag{49}$$

where $\boldsymbol{k}_c$ is the wavevector in comoving coordinates and the proper wavevector $\boldsymbol{k}$ is related to $\boldsymbol{k}_c$ by $\boldsymbol{k}_c = R\boldsymbol{k}$. This is a key equation we have been seeking.

The Jeans' Instability (1)

The differential equation for gravitational instability in a static medium is obtained by setting $\dot{R} = 0$. Then, for waves of the form $\Delta = \Delta_0 \, \exp \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)$, the dispersion relation,

$$\omega^2 = c_\mathrm{s}^2 k^2 - 4\pi G \varrho_0 \,, \tag{50}$$

is obtained.

- If $c_\mathrm{s}^2 k^2 > 4\pi G \varrho_0$, the right-hand side is positive and the perturbations are oscillatory, that is, they are sound waves in which the pressure gradient is sufficient to provide support for the region. Writing the inequality in terms of wavelength, stable oscillations are found for wavelengths less than the critical *Jeans' wavelength* λ_J

$$\lambda_\mathrm{J} = \frac{2\pi}{k_\mathrm{J}} = c_\mathrm{s} \left(\frac{\pi}{G\varrho} \right)^{1/2} \,. \tag{51}$$

The Jeans' Instability (2)

- If $c_s^2 k^2 < 4\pi G \varrho_0$, the right-hand side of the dispersion relation is negative, corresponding to unstable modes. The solutions can be written

$$\Delta = \Delta_0 \exp(\Gamma t + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \, , \tag{52}$$

where

$$\Gamma = \pm \left[4\pi G \varrho_0 \left(1 - \frac{\lambda_J^2}{\lambda^2} \right) \right]^{1/2} \, . \tag{53}$$

The positive solution corresponds to exponentially growing modes. For wavelengths much greater than the Jeans' wavelength, $\lambda \gg \lambda_J$, the growth rate Γ becomes $(4\pi G \varrho_0)^{1/2}$. In this case, the characteristic growth time for the instability is

$$\tau = \Gamma^{-1} = (4\pi G \varrho_0)^{-1/2} \sim (G \varrho_0)^{-1/2} \, . \tag{54}$$

This is the famous *Jeans' Instability* and the time scale τ is the typical collapse time for a region of density ϱ_0.

The Jeans' Instability in an Expanding Medium

We return first to the full version of the differential equation for Δ.

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \Delta(4\pi G\varrho - k^2 c_\mathrm{s}^2)\,. \tag{55}$$

The second term $2(\dot{R}/R)(\mathrm{d}\Delta/\mathrm{d}t)$ modifies the classical Jeans' analysis in crucial ways. It is apparent from the right-hand side of (55) that the Jeans' instability criterion applies in this case also but the growth rate is significantly modified. Let us work out the growth rate of the instability in the long wavelength limit $\lambda \gg \lambda_\mathrm{J}$, in which case we can neglect the pressure term $c_\mathrm{s}^2 k^2$. We therefore have to solve the equation

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{\mathrm{d}\Delta}{\mathrm{d}t} = 4\pi G\varrho_0\Delta\,. \tag{56}$$

Before considering the general solution, let us first consider the special cases $\Omega_0 = 1$ and $\Omega_0 = 0$ for which the scale factor-cosmic time relations are $R = (\frac{3}{2}H_0 t)^{2/3}$ and $R = H_0 t$ respectively.

The Jeans' Instability in an Expanding Medium

The Einstein–de Sitter Critical Model $\Omega_0 = 1$. In this case,

$$4\pi G\varrho = \frac{2}{3t^2} \qquad \text{and} \qquad \frac{\dot{R}}{R} = \frac{2}{3t} \, . \tag{57}$$

Therefore,

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + \frac{4}{3t}\frac{\mathrm{d}\Delta}{\mathrm{d}t} - \frac{2}{3t^2}\Delta = 0 \, . \tag{58}$$

By inspection, it can be seen that there must exist power-law solutions of (58) and so we seek solutions of the form $\Delta = at^n$. Hence

$$n(n-1) + \tfrac{4}{3}n - \tfrac{2}{3} = 0 \, , \tag{59}$$

which has solutions $n = 2/3$ and $n = -1$. The latter solution corresponds to a decaying mode. The $n = 2/3$ solution corresponds to the growing mode we are seeking, $\Delta \propto t^{2/3} \propto R = (1+z)^{-1}$. This is the key result

$$\boxed{\Delta = \frac{\delta\varrho}{\varrho} \propto (1+z)^{-1} \, .} \tag{60}$$

In contrast to the *exponential* growth found in the static case, the growth of the perturbation in the case of the critical Einstein–de Sitter universe is *algebraic*.

The Jeans' Instability in an Expanding Medium

The Empty, Milne Model $\Omega_0 = 0$ In this case,

$$\varrho = 0 \qquad \text{and} \qquad \frac{\dot{R}}{R} = \frac{1}{t}\,, \tag{61}$$

and hence

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + \frac{2}{t}\frac{\mathrm{d}\Delta}{\mathrm{d}t} = 0\,. \tag{62}$$

Again, seeking power-law solutions of the form $\Delta = at^n$, we find $n = 0$ and $n = -1$, that is, there is a decaying mode and one of constant amplitude Δ = constant.

These simple results describe the evolution of small amplitude perturbations, $\Delta = \delta\varrho/\varrho \ll 1$. In the early stages of the matter-dominated phase, the dynamics of the world models approximate to those of the Einstein–de Sitter model, $R \propto t^{2/3}$, and so the amplitude of the density contrast grows linearly with R. In the late stages at redshifts $\Omega_0 z \ll 1$, when the Universe may approximate to the $\Omega_0 = 0$ model, the amplitudes of the perturbations grow very slowly and, in the limit $\Omega_0 = 0$, do not grow at all.

The General Solutions

A general solution for the growth of the density contrast with scale-factor for all pressure-free Friedman world models can be rewritten in terms of the density parameter Ω_0 as follows:

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \frac{3\Omega_0 H_0^2}{2} R^{-3}\Delta \,, \qquad (63)$$

where, in general,

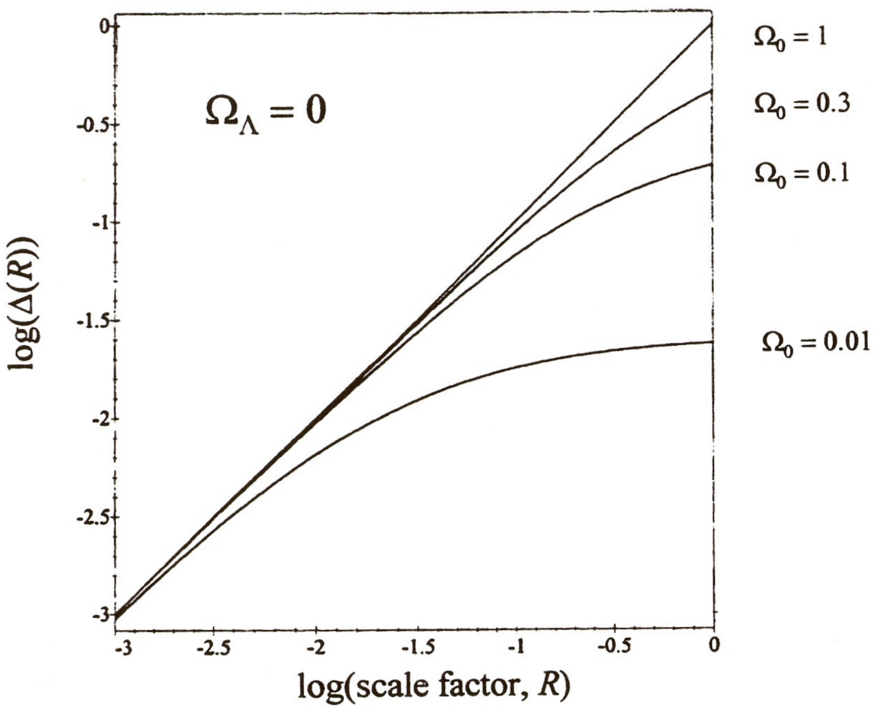
$$\dot{R} = H_0\left[\Omega_0\left(\frac{1}{R} - 1\right) + \Omega_\Lambda(R^2 - 1) + 1\right]^{1/2} . \qquad (64)$$

The solution for the growing mode can be written as follows:

$$\Delta(R) = \frac{5\Omega_0}{2}\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)\int_0^R \frac{\mathrm{d}R'}{\left(\frac{\mathrm{d}R'}{\mathrm{d}t}\right)^3} \,, \qquad (65)$$

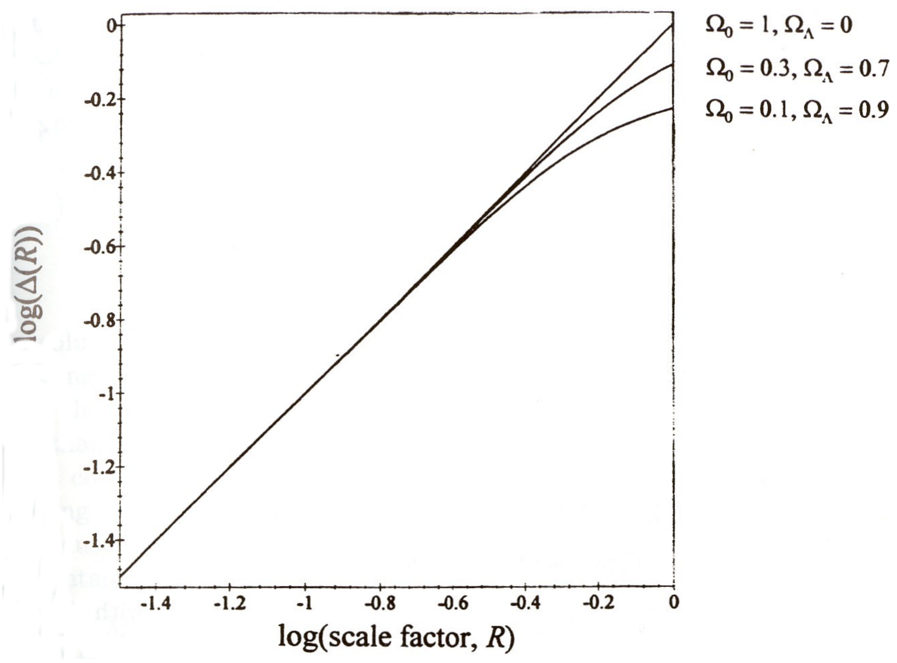
where the constants have been chosen so that the density contrast for the standard critical world model with $\Omega_0 = 1$ and $\Omega_\Lambda = 0$ has unit amplitude at the present epoch, $R = 1$. With this scaling, the density contrasts for all the examples we will consider correspond to $\Delta = 10^{-3}$ at $R = 10^{-3}$. It is simplest to carry out the calculations numerically for a representative sample of world models.

Models with $\Omega_\Lambda = 0$



The development of density fluctuations from a scale factor $R = 1/1000$ to $R = 1$ are shown for a range of world models with $\Omega_\Lambda = 0$. These results are consistent with the calculations carried out above, in which it was argued that the amplitudes of the density perturbations vary as $\Delta \propto R$ so long as $\Omega_0 z \gg 1$, but the growth essentially stops at smaller redshifts.

Models with finite Ω_Λ



The models of greatest interest are the flat models for which $(\Omega_0 + \Omega_\Lambda) = 1$, in all cases, the fluctuations having amplitude $\Delta = 10^{-3}$ at $R = 10^{-3}$. The growth of the density contrast is somewhat greater in the cases Ω_0 = 0.1 and 0.3 as compared with the corresponding cases with $\Omega_\Lambda = 0$. The fluctuations continue to grow to greater values of the scale-factor R, corresponding to smaller redshifts, as compared with the models with $\Omega_\Lambda = 0$.

The Relativistic Case

In the radiation-dominated phase of the Big Bang, the primordial perturbations are in a radiation-dominated plasma, for which the relativistic equation of state $p = \frac{1}{3}\varepsilon$ is appropriate.

The equation of energy conservation becomes

$$\frac{\partial \varrho}{\partial t} = -\nabla \cdot \left(\varrho + \frac{p}{c^2}\right) \boldsymbol{v}; \qquad (66)$$

$$\frac{\partial}{\partial t}\left(\varrho + \frac{p}{c^2}\right) = \frac{\dot{p}}{c^2} - \left(\varrho + \frac{p}{c^2}\right)(\nabla \cdot \boldsymbol{v})\,. \qquad (67)$$

Substituting $p = \frac{1}{3}\varrho c^2$ into (66) and (67), the relativistic continuity equation is obtained:

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -\frac{4}{3}\varrho(\nabla \cdot \mathrm{v})\,. \qquad (68)$$

Euler's equation for the acceleration of an element of the fluid in the gravitational potential ϕ becomes

$$\left(\varrho + \frac{p}{c^2}\right)\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right] = -\nabla p - \left(\varrho + \frac{p}{c^2}\right)\nabla \phi\,. \qquad (69)$$

The Relativistic Case

If we neglect the pressure gradient term, (69) reduces to the familiar equation

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla\phi \, . \tag{70}$$

Finally, the differential equation for the gravitational potential ϕ becomes

$$\nabla^2\phi = 4\pi G\left(\varrho + \frac{3p}{c^2}\right) \, . \tag{71}$$

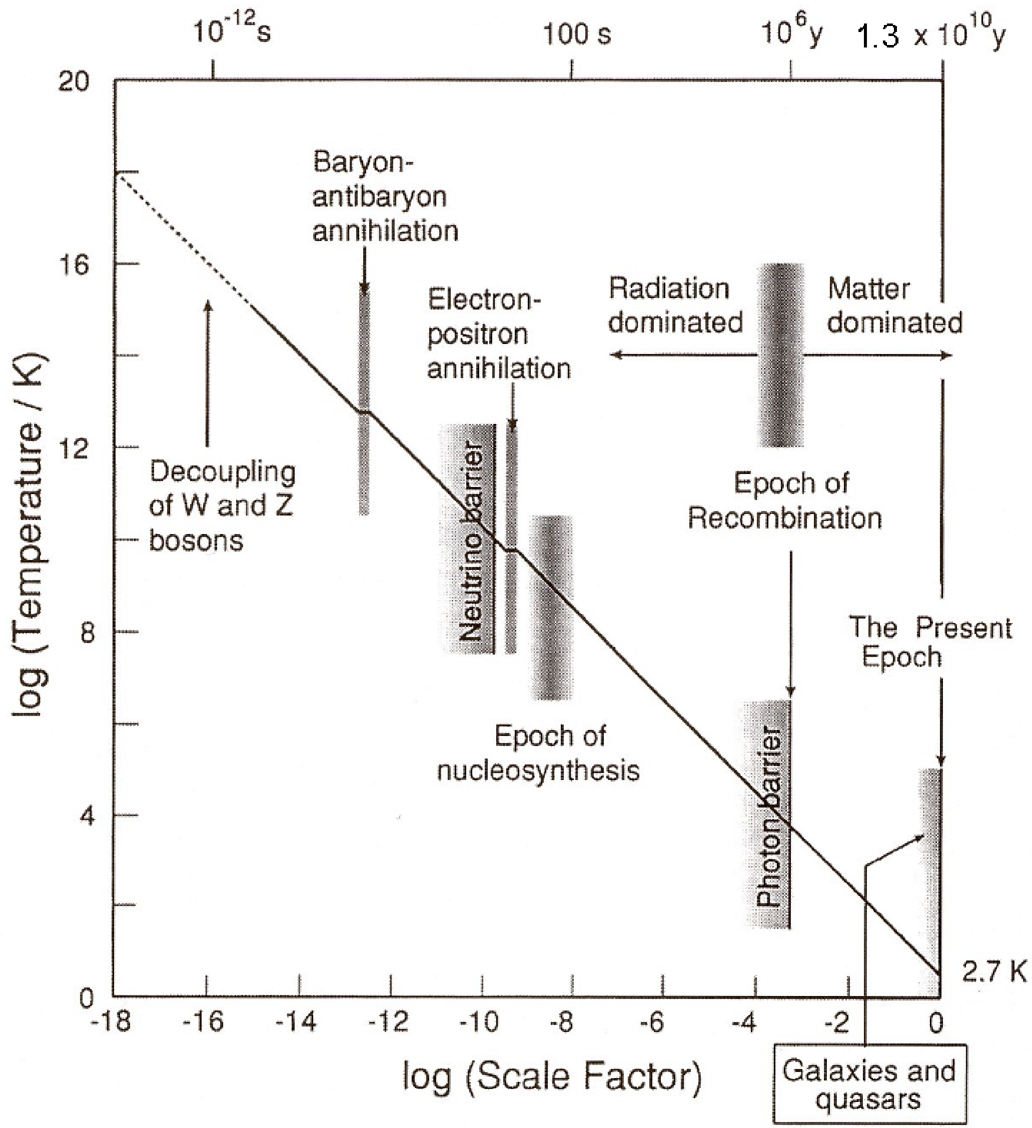
For a fully relativistic gas, $p = \frac{1}{3}\varrho c^2$ and so

$$\nabla^2\phi = 8\pi G\varrho \, . \tag{72}$$

The net result is that the equations for the evolution of the perturbations in a relativistic gas are of similar mathematical form to the non-relativistic case. The same type of analysis which was carried out above leads to the following equation

$$\boxed{\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \Delta\left(\frac{32\pi G\varrho}{3} - k^2 c_s^2\right) \, .} \tag{73}$$

Summary of the Thermal History of the Universe



This diagram summarises the key epochs in the thermal history of the Universe. The key epochs are

- The epoch of recombination.
- The epoch of equality of matter and radiation.

The Radiation Dominated Era

At redshifts $z \gg 4 \times 10^4 \Omega_0 h^2$, the Universe was radiation-dominated. If we take into account the contribution of the neutrinos as well, the expression becomes $\varepsilon = 1.68 a T_{\mathrm{rad}}^4$ and so massless particles dominate the dynamics of the Universe at redshifts

$$z \geq 2.4 \times 10^4 \Omega_0 h^2 = 3,500$$

for the concordance values of the parameters.

If the matter and radiation were not thermally coupled, they would cool independently, the hot gas having ratio of specific heats $\gamma = 5/3$ and the radiation $\gamma = 4/3$. These result in adiabatic cooling which depends upon the scale factor R as $T_{\mathrm{B}} \propto R^{-2}$ and $T_{\mathrm{r}} \propto R^{-1}$ for the diffuse baryonic matter and radiation respectively. This is not the case, however, during the pre-recombination and immediate post-recombination eras because the matter and radiation are strongly coupled by *Compton scattering*. The optical depth of the pre-recombination plasma for Thomson scattering is very large, so large that we can no longer ignore the small energy transfers which take place between the photons and the electrons in Compton collisions.

The Sound Speed as a Function of Cosmic Epoch

All sound speeds are proportional to the square root of the ratio of the pressure which provides the restoring force to the inertial mass density of the medium. The speed of sound c_{s} is given by

$$c_{\mathrm{s}}^2 = \left(\frac{\partial p}{\partial \varrho}\right)_S \ , \tag{74}$$

where the subscript S means ‘at constant entropy’, that is, we consider adiabatic sound waves. From the epoch when the energy densities of matter and radiation were equal to beyond the epoch of recombination, the dominant contributors to p and ϱ change dramatically as the Universe changes from being radiation- to matter-dominated. The sound speed can then be written

$$c_{\mathrm{s}}^2 = \frac{(\partial p/\partial T)_{\mathrm{r}}}{(\partial \varrho/\partial T)_{\mathrm{r}} + (\partial \varrho/\partial T)_{\mathrm{m}}}, \tag{75}$$

where the partial derivatives are taken at constant entropy. It is straightforward to show that this reduces to the following expression:

$$c_{\mathrm{s}}^2 = \frac{c^2}{3}\frac{4\varrho_{\mathrm{r}}}{4\varrho_{\mathrm{r}} + 3\varrho_{\mathrm{m}}} \ . \tag{76}$$

The Damping of Sound Waves

Although the matter and radiation are closely coupled throughout the pre-recombination era, the coupling is not perfect and radiation can diffuse out of the density perturbations. Since the radiation provides the restoring force for support for the perturbation, the perturbation is damped out if the radiation has time to diffuse out of it. This process is often referred to as *Silk damping*.

At any epoch, the mean free path for scattering of photons by electrons is $\lambda = (N_{\rm e}\sigma_{\rm T})^{-1}$, where $\sigma_{\rm T} = 6.665 \times 10^{-29}$ m^2 is the Thomson cross-section. The distance which the photons can diffuse is

$$r_{\rm D} \approx (Dt)^{1/2} = \left(\tfrac{1}{3}\lambda c t\right)^{1/2} , \qquad (77)$$

where t is cosmic time. The baryonic mass within this radius, $M_{\rm D} = (4\pi/3) r_{\rm D}^3 \varrho_{\rm B}$, can now be evaluated for the pre-recombination era.

Horizons and the Horizon Problem

One of the key concepts is that of *particle horizons*. At any epoch t, the particle horizon is defined to be the distance a light signal could have travelled from the origin of the Big Bang at $t = 0$ by the epoch t. Its value is

$$r_{\mathrm{H}}(t) = R(t) \int_0^t \frac{c\,\mathrm{d}t}{R(t)} = \frac{1}{1+z} \int_\infty^z (1+z) c\,\mathrm{d}t \,. \tag{78}$$

At early times, all the Friedman models tend toward the dynamics of the critical model

$$R = \Omega_0^{1/3} \left(\frac{3H_0 t}{2}\right)^{2/3} ,$$

and so the particle horizon becomes $r_{\mathrm{H}}(t) = 3ct$.

A similar calculation can be carried out for the radiation-dominated era shows that $r_{\mathrm{H}}(t) = 2ct$.

The Horizon Problem

We can now use these results to illustrate the origin of the *horizon problem* for the standard Friedman models with $\Omega_\Lambda = 0$. The particle horizon on the last scattering surface subtends an angle θ_H according to an observer at the present epoch. At a redshift $z = 1000$, we can safely use the standard matter-dominated solutions of Friedman's equation in the limit $\Omega_0 z \gg 1$, $D = 2c/H_0\Omega_0$ and so

$$\theta_\mathrm{H} = \frac{r_\mathrm{H}(t)(1+z)}{D} = \frac{\Omega_0^{1/2}}{(1+z)^{1/2}} = 1.8\Omega_0^{1/2} \quad \text{degrees} . \tag{79}$$

This result means that, according to the standard Friedman picture, regions of the Universe separated by an angle of more than

$$1.8\Omega_0^{1/2} \text{ degrees}$$

on the sky could not have been in causal contact on the last scattering surface. Why then is the Cosmic Microwave Background Radiation so uniform over the whole sky to a precision of about one part in 10^5?

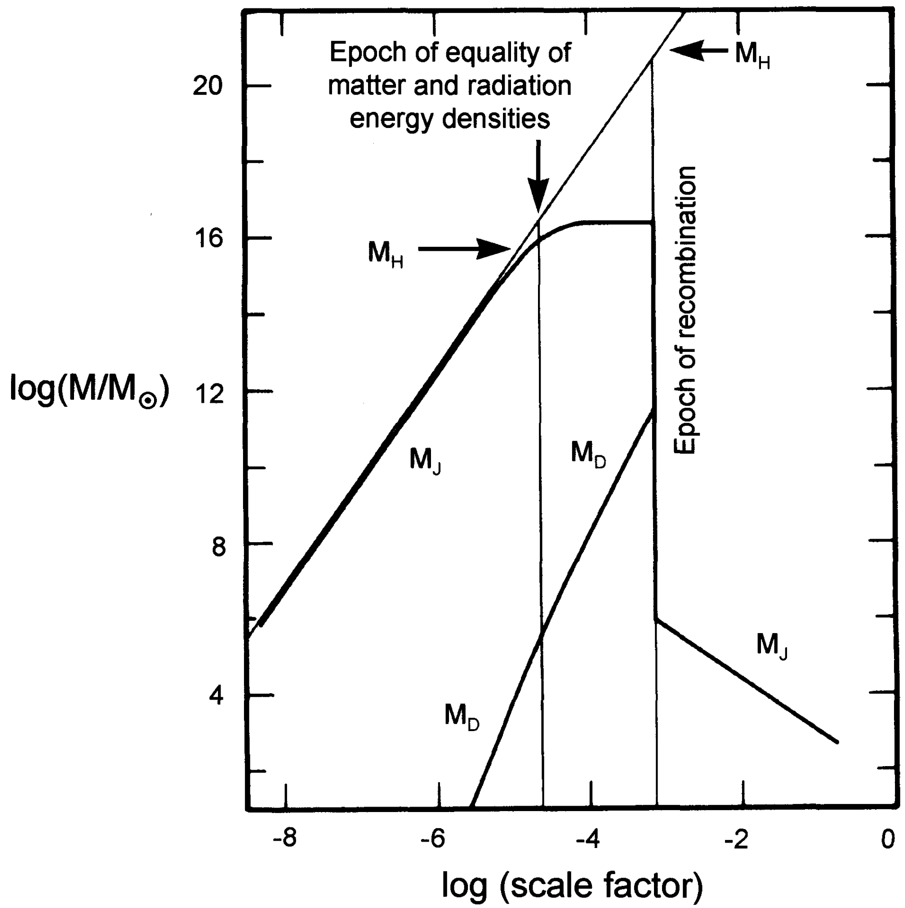
The Inflationary Solution

This horizon problem is circumvented in the inflationary model of the very early Universe because of the exponential expansion of the scale factor which ensures that opposite directions on the sky were in causal contact. To illustrate this, consider the de Sitter model. Normalising $R(t)$ to the value unity at the present epoch, we find

$$\begin{aligned} R(t) &= \exp\,[\Omega_\Lambda^{1/2} H_0 (t - t_0)], \\ r_{\mathrm{H}}(t) &= \frac{c}{H_0} R(t) \frac{\left[\exp\,(\Omega_\Lambda^{1/2} H_0 t) - 1\right]}{\Omega_\Lambda^{1/2}}\,. \end{aligned} \tag{80}$$

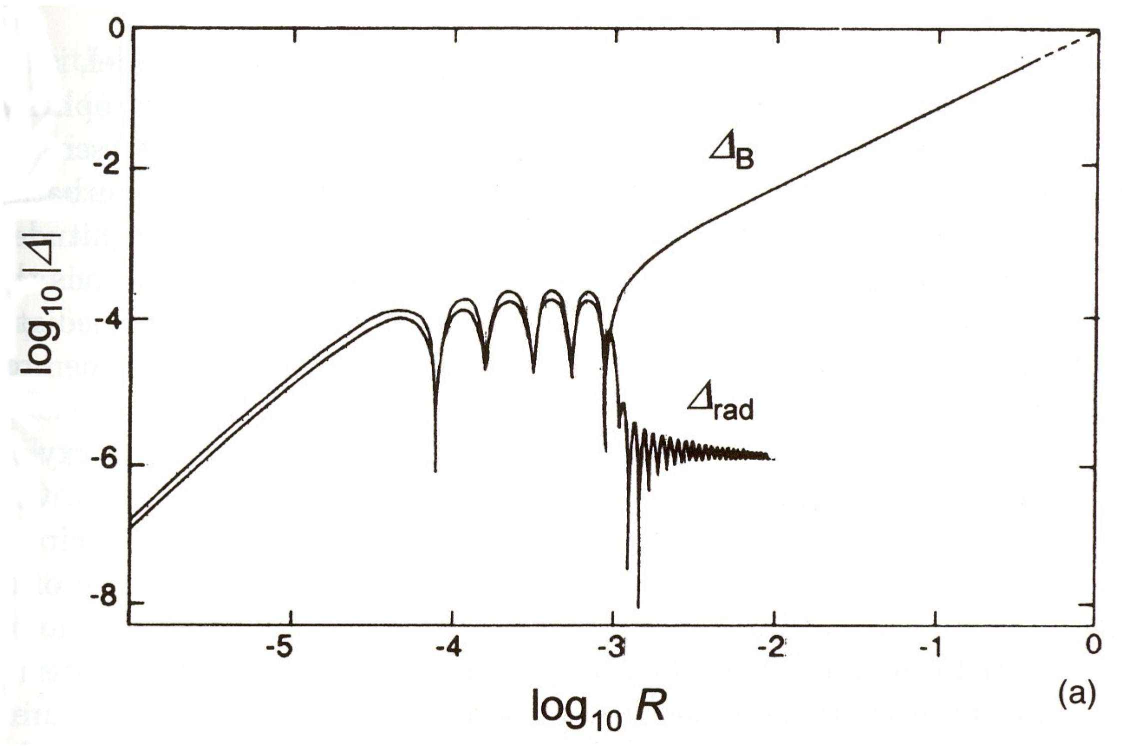
In the inflationary picture, the value of $\Omega_\Lambda^{1/2}$ is enormous and so causal communication can extend far beyond the scale ct when $\Omega_\Lambda^{1/2} H_0 t \gg 1$.

The Simple Baryonic Picture



We can put together all these ideas to develop the simplest picture of galaxy formation. This is the simplest baryonic picture. It includes many of the features which will reappear in the ΛCDM picture. The diagram shows how the horizon mass M_{H}, the Jeans mass M_{J} and the Silk Mass M_{D} change with scale factor R.

The Simple Baryonic Picture

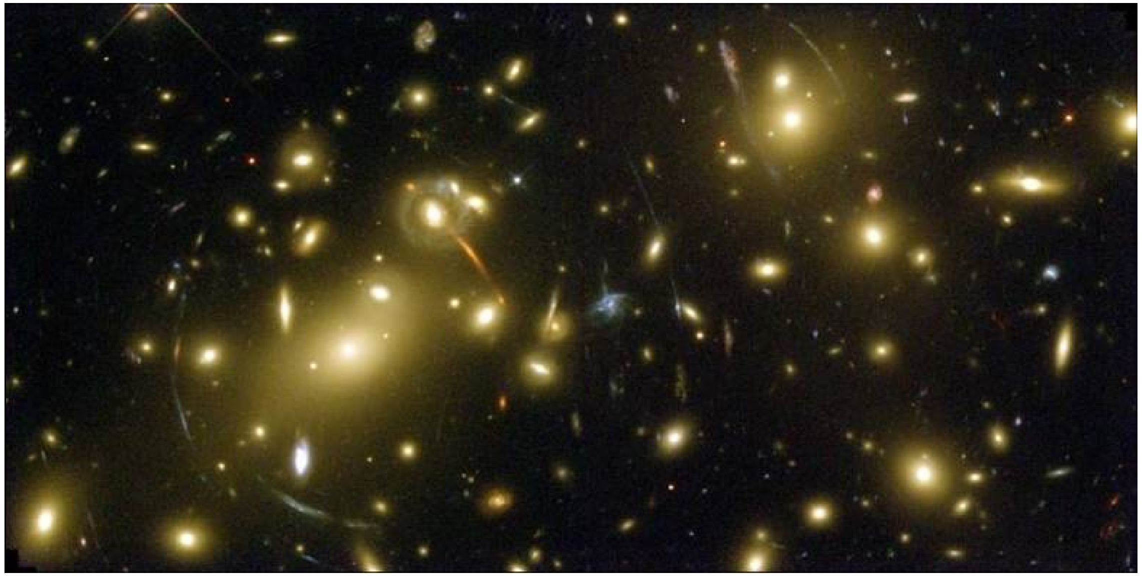


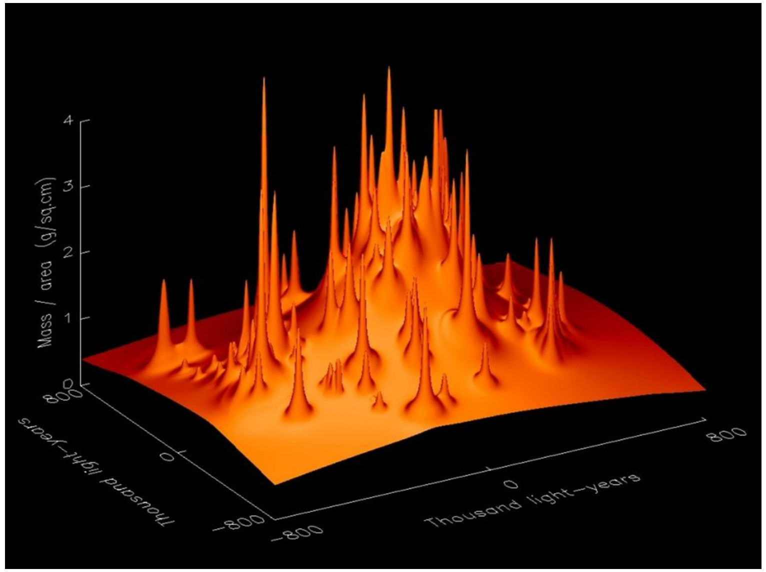
This diagrams, from Coles and Lucchin (1995) shows schematially how structure develops in a purely baryonic Universe. The problem is that the temperature fluctuations on the last scattering surface as expected to be at least $\Delta T/T \sim 10^{-3}$, far in excess of the observed limits.

The solution to this problem came with the realisation that the dark matter is the dominant contribution to Ω_0.

Dark Matter

There is no question but that the Universe is dominated gravitationally on small scales by Dark Matter.





These reconstructions of the total mass distribution from gravitational lensing show that the dark matter is dynamically dominant in clusters of galaxies.

Instabilities in the Presence of Dark Matter

Neglecting the internal pressure of the fluctuations, the expressions for the density contrasts in the baryons and the dark matter, Δ_{B} and Δ_{D} respectively, can be written as a pair of coupled equations

$$\ddot{\Delta}_{\mathrm{B}} + 2\left(\frac{\dot{R}}{R}\right)\dot{\Delta}_{\mathrm{B}} = A\varrho_{\mathrm{B}}\Delta_{\mathrm{B}} + A\varrho_{\mathrm{D}}\Delta_{\mathrm{D}}\,, \tag{81}$$

$$\ddot{\Delta}_{\mathrm{D}} + 2\left(\frac{\dot{R}}{R}\right)\dot{\Delta}_{\mathrm{D}} = A\varrho_{\mathrm{B}}\Delta_{\mathrm{B}} + A\varrho_{\mathrm{D}}\Delta_{\mathrm{D}}\,. \tag{82}$$

Let us find the solution for the case in which the dark matter has $\Omega_0 = 1$ and the baryon density is negligible compared with that of the dark matter. Then (82) reduces to the equation for which we have already found the solution $\Delta_{\mathrm{D}} = BR$ where B is a constant. Therefore, the equation for the evolution of the baryon perturbations becomes

$$\ddot{\Delta}_{\mathrm{B}} + 2\left(\frac{\dot{R}}{R}\right)\dot{\Delta}_{\mathrm{B}} = 4\pi G\varrho_{\mathrm{D}}BR\,. \tag{83}$$

Instabilities in the Presence of Dark Matter

Since the background model is the critical model for which $R = (3H_0t/2)^{2/3}$ and $3H_0^2 = 8\pi G\varrho_{\rm D}$, equation (83) simplifies to

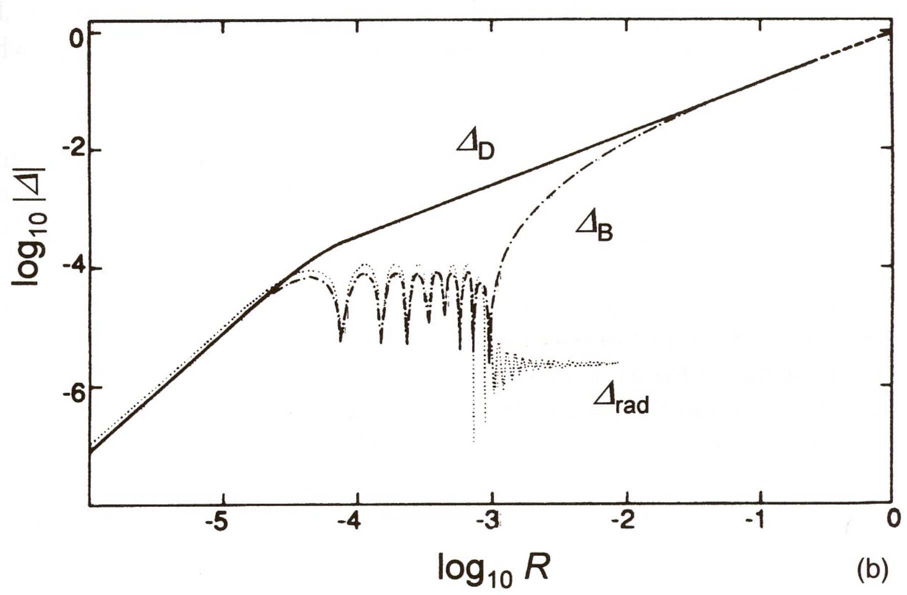
$$R^{3/2}\frac{\rm d}{{\rm d}R}\left(R^{-1/2}\frac{{\rm d}\Delta}{{\rm d}R}\right) + 2\frac{{\rm d}\Delta}{{\rm d}R} = \tfrac{3}{2}B \,. \tag{84}$$

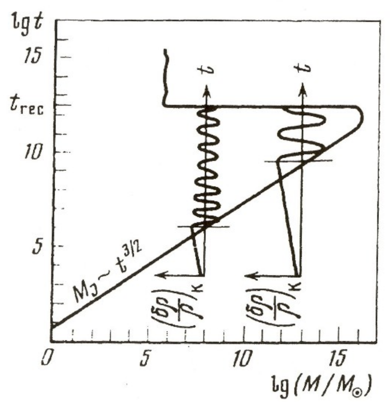
The solution, $\Delta = B(R - R_0)$, satisfies (84). This result has the following significance. Suppose that, at some redshift z_0, the amplitude of the baryon fluctuations is very small, that is, very much less than that of the perturbations in the dark matter. The above result shows how the amplitude of the baryon perturbation develops subsequently under the influence of the dark matter perturbations. In terms of redshift we can write

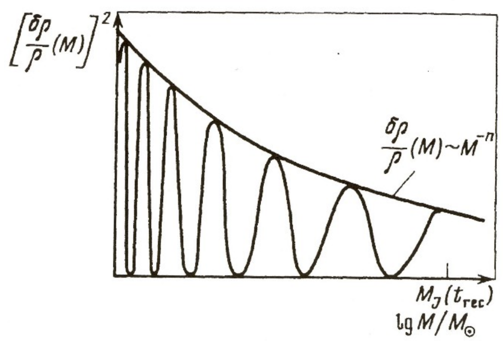
$$\Delta_{\rm B} = \Delta_{\rm D}\left(1 - \frac{z}{z_0}\right) \,. \tag{85}$$

Thus, the amplitude of the perturbations in the baryons grows rapidly to the same amplitude as that of the dark matter perturbations. To put it crudely, the baryons fall into the dark matter perturbations and, within a factor of two in redshift, have amplitudes half that of the dark matter perturbations.

The Cold Dark Matter Picture







This diagrams, also from Coles and Lucchin (1995) shows schematially how structure develops in a cold dark matter dominated Universe. Notice how the amplitudes of the baryonic perturbations were very much smaller than those in the cold dark matter.

Note also the origin of the Acoustic or Sakharov peaks in the predicted mass spectrum (from Sunyaev and Zeldovich 1970).

This is the favoured model for the formation structure.

The Input Parameters for the Models

- Selection of a cosmological model with values of Ω_0, Ω_Λ and H_0.

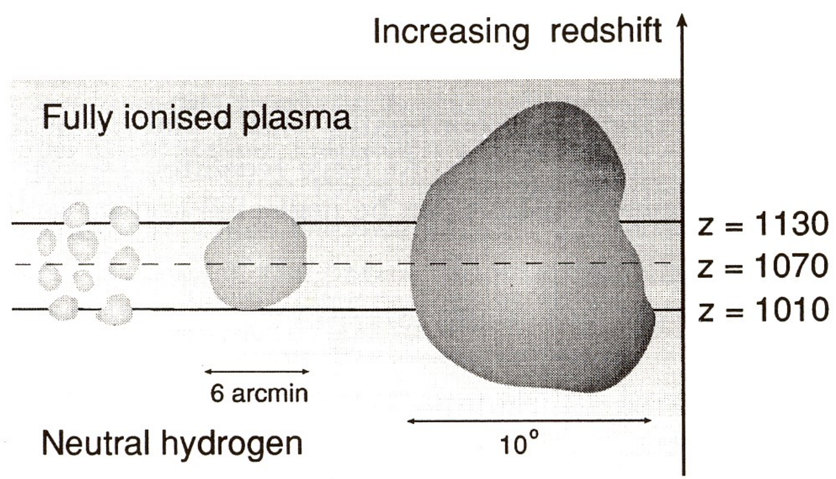
- The ordinary baryonic matter has density parameter Ω_B, which is only about 5-10% of the dark matter.

- The power-spectrum of the initial perturbations is assumed to be of Harrison-Zeldovich form $p(k) = Ak^n$ with random phases. The value of n can be varied to find the best fit to the observations.

Many other components can be included.

Show simulations.

Perturbations on the Last Scattering Layer



The diagram shows the range of redshifts between which half of the photons of the CMB were last scattered.

The diagram shows schematically the size of various small perturbations compared with the thickness of the last scattering layer. On very large scales, the perturbations are very much larger than the thickness of the layer. On scales less than clusters of galaxies, many perturbations overlap, reducing the amplitude of the perturbations.

Large Angular Scales - the Sachs-Wolfe Effect

On the very largest scales, the dominant source of intensity fluctuations results from the fact that the photons we observe have to climb out of the gravitational potential wells associated with perturbations which are very much greater in size than the thickness of the last scattering layer.

On the scales of interest, the fluctuations at the epoch of recombination far exceed the horizon scale and so the perturbations would represent a change of the gravitational potential of everything within the horizon. More properly, we should describe these perturbations as *metric perturbations*. These ‘super-horizon’ perturbations raise the thorny question of the choice of gauge to be used in relativistic perturbation theory. A general relativistic treatment, first performed by Sachs and Wolfe (1967), is needed. The result is $\Delta T/T = (1/3)\Delta\phi/c^2$, recalling that $\Delta\phi$ is a negative quantity.

The Sachs-Wolfe Effect
The Coles-Lucchin Argument

Coles and Lucchin (1995) rationalised how the Sachs–Wolfe answer can be found. In addition to the Newtonian gravitational redshift, because of the perturbation of the metric, the cosmic time, and hence the scale factor R, at which the fluctuations are observed, are shifted to slightly earlier cosmic times. Temperature and scale factor change as $\Delta T/T = -\Delta R/R$. For all the standard models in the matter-dominated phase $R \propto t^{2/3}$ and so the increment of cosmic time changes as $\Delta R/R = (2/3)\Delta t/t$.

But $\Delta\nu/\nu = -\Delta t/t$ is just the Newtonian gravitational redshift, with net result that there is a positive contribution to $\Delta T/T$ of $-(2/3)\Delta\phi/c^2$. The net temperature fluctuation is $\Delta T/T = \frac{1}{3}\Delta\phi/c^2$.

It is then a straightforward calculation to show that, for the $\Omega_0 = 1$ model, the temperature fluctuations depend upon angular scale as

$$\frac{\Delta T}{T} \approx \frac{1}{3}\frac{\Delta\phi}{c^2} \propto \theta^{(1-n)/2} \; . \tag{86}$$

Intermediate Angular Scales

The first acoustic peak is associated with perturbations on the scale of the sound horizon at the epoch of recombination. The amplitudes of the acoustic waves at the last scattering layer depend upon the phase difference from the time they came through the horizon to last scattering layer, that is, they depend upon

$$\int \mathrm{d}\phi = \int \omega \, \mathrm{d}t \, . \tag{87}$$

Let us label the wavenumber of the first acoustic peak k_1. Oscillations which are $n\pi$ out of phase with the first acoustic peak also correspond to maxima in the temperature power spectrum at the epoch of recombination. There is, however, an important difference between the even and odd harmonics of k_1. The odd harmonics correspond to the maximum compression of the waves and so to increases in the temperature, whereas the even harmonics correspond to rarefactions of the acoustic waves and so to temperature minima. The perturbations with phase differences $\pi(n + \frac{1}{2})$ relative to that of the first acoustic peak have zero amplitude at the last scattering layer and correspond to the minima in the power spectra.

Intermediate Angular Scales

To find the acoustic peaks, we need to find the wavelengths corresponding to frequencies

$$\omega t_{\mathrm{rec}} = n\pi \; . \tag{88}$$

Adopting the short wavelength dispersion relation ,

$$\omega^2 = c_{\mathrm{s}}^2 k^2 - 4\pi G \varrho_{\mathrm{B}} = c_{\mathrm{s}}^2 (k^2 - k_{\mathrm{J}}^2) \approx c_{\mathrm{s}}^2 k^2 \; , \tag{89}$$

the condition becomes

$$c_{\mathrm{s}} k_n t_{\mathrm{rec}} = n\pi \qquad k_n = \frac{n\pi}{\lambda_{\mathrm{s}}} = n k_1 \; . \tag{90}$$

Thus, the acoustic peaks are expected to be roughly evenly spaced in wavenumber. The separation between the acoustic peaks thus provides us with further information about various combinations of cosmological parameters.

Intermediate Angular Scales

The next task is to determine the amplitudes of the acoustic peaks in the power spectrum. The complication is that the acoustic oscillations take place in the presence of growing density perturbations in the dark matter, which have greater amplitude than those in the acoustic oscillations. Therefore, in dark matter scenarios, the acoustic waves are driven by the larger density perturbations in the dark matter with the same wavelength, that is, the perturbations are forced oscillations. In a simple approximation, growth rate of the oscillation is driven by the growing amplitude of the dark matter perturbations:

$$\frac{\mathrm{d}^2\Delta_\mathrm{B}}{\mathrm{d}t^2} = \Delta_\mathrm{D} 4\pi G\rho_D - \Delta_\mathrm{B} k^2 c_\mathrm{s}^2 \, . \tag{91}$$

The sound speed is given by

$$c_\mathrm{s} = \frac{c}{\sqrt{3}}\left(\frac{4\varrho_\mathrm{rad}}{4\varrho_\mathrm{rad} + 3\varrho_\mathrm{B}}\right)^{1/2} = \frac{c}{\sqrt{3(1+\mathcal{R})}} \, , \tag{92}$$

Intermediate Angular Scales

In the limit $\mathcal{R} \to 0$, the monopole and dipole temperature fluctuations are of the same amplitude. However, when the inertia of the baryons can no longer be neglected, the monopole contribution becomes significantly greater than the dipole term.

At maximum compression, $k\lambda_{\mathrm{S}} = \pi$, the amplitude of the observed temperature fluctuation is $(1 + 6\mathcal{R})$ times that of the Sachs–Wolfe effect. Furthermore, the amplitudes of the oscillations are asymmetric if $\mathcal{R} \neq 0$, the temperature excursions varying between $-(\Psi/c^2)(1 + 6\mathcal{R})$ for $k\lambda_{\mathrm{S}} = (2n + 1)\pi$ and (Ψ/c^2) for $k\lambda_{\mathrm{S}} = 2n\pi$.

These results can account for the some of the prominent features of the temperature fluctuation spectrum. The temperature perturbations associated with the acoustic peaks are much larger than the Sachs–Wolfe fluctuations. The asymmetry between the even and odd peaks in the fluctuation spectrum is associated with the extra compression at the bottom of the gravitational potential wells when account is taken of the inertia of the perturbations associated with the baryonic matter.

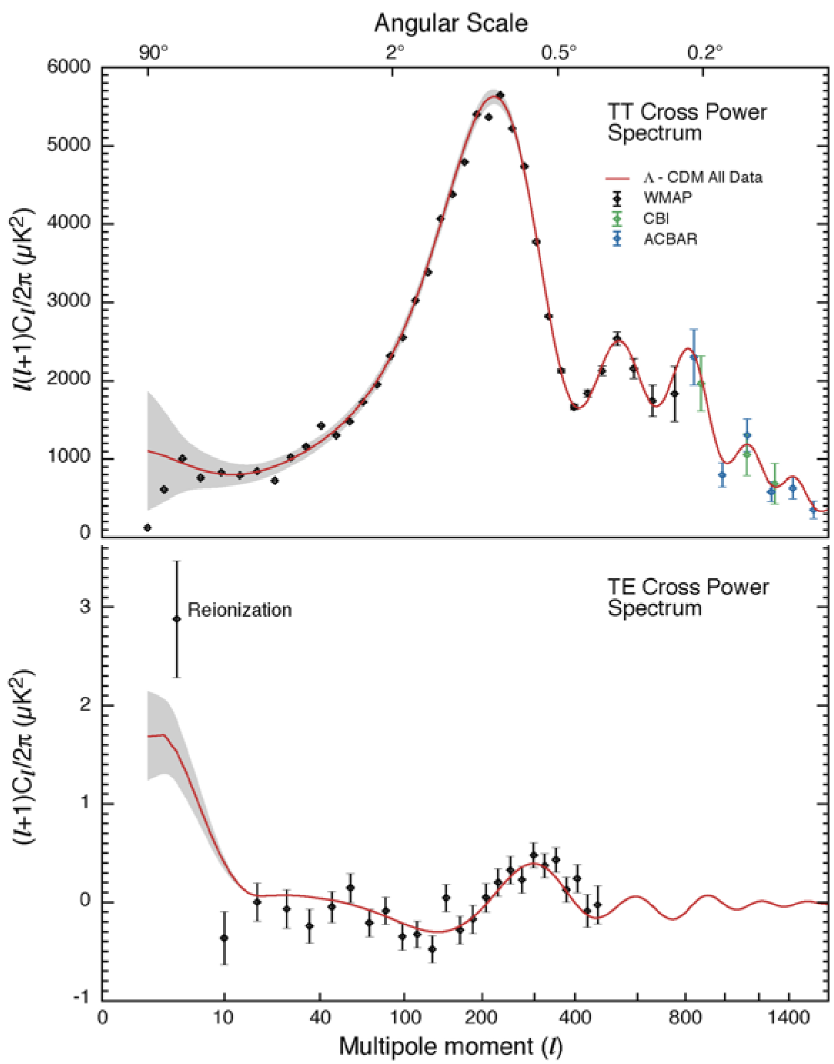
Small Angular Scales

- Silk Damping scale results in the suppression of high wave number modes on scales less than about 8 Mpc at the present epoch.

- The superposition of perturbations damps out the perturbations within the last scattering layer.

- The Sunyaev-Zeldovich effect associated with hot intergalactic gas in clusters of galaxies creates additional small scale perturbations.

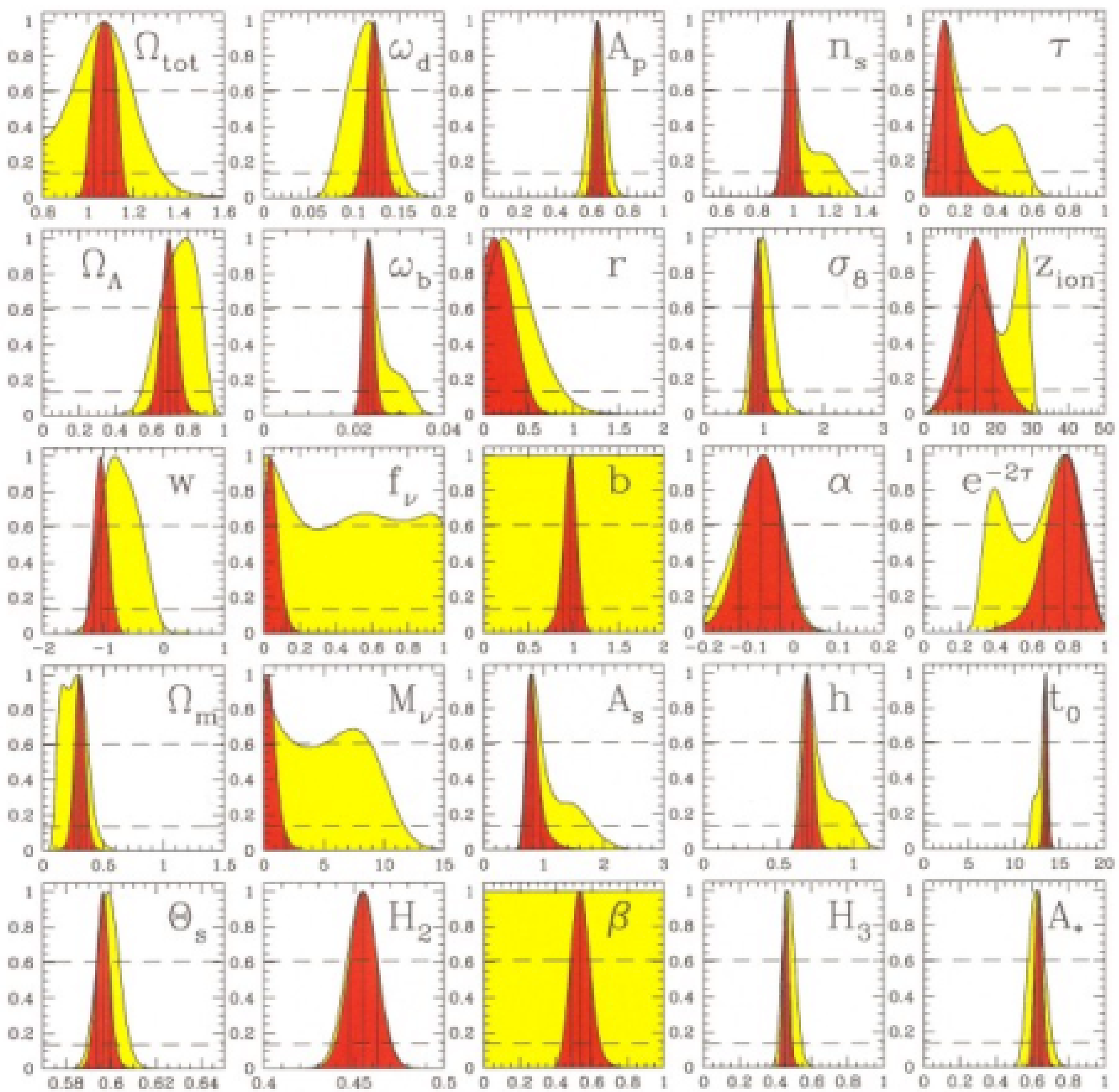
The WMAP Power Spectrum



Many of the features of the above analysis can be observed in the WMAP power spectrum.

- The location of the maximum of the first peak in the power spectrum.
- The asymmetry between the first, second and third peaks.
- The flatness of the spectrum at low values of l.
- The polarisation and the large signal at very small values of l.

Parameter Estimation using WMAP and SDSS



Max Tegmark and his colleagues have used the WMAP power-spectrum and polarisation to make parameter estimates. The yellow areas show probability distributions using WMAP alone; the red areas include the power spectrum of galaxies from the Sloan Digital Sky Survey.

Parameter Estimation using WMAP and SDSS

Parameter	Status	WMAP alone	WMAP + SDSS
$\omega_{\mathrm{B}} = \Omega_{\mathrm{B}} h^2$	Not optional	$0.0245^{+0.0050}_{-0.0019}$	$0.0232^{+0.0013}_{-0.0010}$
$\omega_{\mathrm{D}} = \Omega_{\mathrm{D}} h^2$	Not optional	$0.115^{+0.020}_{-0.021}$	$0.1222^{+0.0090}_{-0.0082}$
Ω_Λ	Not optional	$0.75^{+0.10}_{-0.10}$	$0.699^{+0.042}_{-0.045}$
w			
τ	Not optional	$0.21^{+0.24}_{-0.11}$	$0.124^{+0.083}_{-0.057}$
$\Omega_k{}^*$			
A_{s}	Not optional	$0.98^{+0.56}_{-0.21}$	$0.81^{+0.15}_{-0.09}$
n_{s}		$1.02^{+0.16}_{-0.06}$	$0.977^{+0.039}_{-0.025}$
α			
r			
n_{t}			
b	Not optional	No constraint	$1.009^{+0.073}_{-0.083}$
$f_\nu = \rho_\nu/\rho_{\mathrm{D}}$			

Concordance Values of the Cosmological Parameters

Parameter	Definition	Value
H_0	Hubble's constant	72 km s^{-1} Mpc^{-1}
Ω_k	space curvature	0
Ω_Λ	dark energy density parameter	0.72
$\Omega_0 = \Omega_B + \Omega_D$	total matter density parameter	0.28
Ω_B	baryon density parameter	0.047
Ω_D	dark matter density parameter	0.233
n_S	scalar spectral index	1
A_S	amplitude of scalar power-spectrum	0.89
τ	reionisation optical depth	0.17

These values agree with independent estimates of these parameters by totally different procedures.

Independent Estimates of Cosmological Parameters

- Hubble Space Telescope estimate of Hubble's constant $h = 72 \pm 7$ km s^{-1} Mpc^{-1}.
- Estimates of Ω_Λ from Type1A supernovae, $\Omega_\Lambda \approx 0.7$.
- Average Mass Density in the Universe from Infall into Superclusters: $\Omega_{\mathrm{m}} = 0.3$ if $h = 0.7$.
- Synthesis of the light elements: $\omega_{\mathrm{b}} = 0.022 \pm 0.002$.
- Nucleocosmochronology: The best estimate of the age of the Galaxy is $T_{\mathrm{gal}} = 12 \pm 2$ billion years.
- Ages of Globular Clusters $T \approx 13$ billion years.